The several faces and the interlaced roots of the "New Algebra"

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Abstract

Who compares Euler's algebra with that of al-Khwārizmī will see more differences than kinship. Al-Khwārizmī uses natural language, while Euler calculates within the syntax of algebraic symbolism. Al-Khwārizmī has a single unknown (a word), Euler as many as he needs, represented by non-linguistic signs. Al-Khwārizmī deals with the unknown and its second power, Euler knows no limits. Al-Khwārizmī's coefficients are numerically fixed, those of Euler may have undetermined values. When al-Khwārizmī operates on a composite expression, he needs roundabout ways; Euler has the parenthesis. The parenthesis mostly goes unmentioned when the characteristics of the "new algebra" are discussed. The rest is familiar. However, the unfolding of the various characteristics is largely left in the dark. The whole seems to have emerged fully grown from the minds of Viète and Descartes. The aim of the paper is to trace the emergence of the various characteristic features of the New Algebra from the 14th-century beginnings of abbacus algebra. The process is far from linear – before the arrival of German *coß* we cannot even speak of "stops and goes" on the road toward some aim. After Christoph Rudolff we probably can; in this final phase, the mostly neglected roles of Michael Stifel and Valentin Mennher are taken up.

Keywords

Descartes, Viète, abbacus algebra, Rechenmeister algebra, Stifel, Mennher, humanist mathematics

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Algebras

"Algebras" constitute what Ludwig Wittgenstein would call *a family* (by commentators often expanded into "natural family") – held together not by a definition but by "a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities, sometimes similarities of detail."[1] It encompasses B. L. van der Waerden's *Moderne Algebra*, built on the work of Emmy Noether and Emil Artin; that of René Descartes and François Viète; furthermore, German *coß* of the 16th century, and Italian abbacus algebra with the culmination brought about by Girolamo Cardano and Nicolò Tartaglia – and of course Arabic *al-jabr*, which gave the family its name; not to speak of Diophantos and algebra-like techniques in Sanskrit and classical Chinese mathematics. Disagreements exist whether the techniques of a single and a double false position or the Old Babylonian area technique are to be counted as algebra – Wittgensteinian families have no clear boundary.

The "new algebra" of my title (henceforth capitalized) is that which was created between *ca* 1600 and *ca* 1700 – by Viète, Descartes, John Wallis and others. At a time when few earlier sources were studied at least by those (few) who were interested in the matter – say, al-Khwārizmī's algebra, Leonardo Fibonacci's *Liber abbaci* (supposed to be more original and more influential than it actually was), Cardano's *Ars magna* and Tartaglia's *General trattato* – this really seemed to be a *new* discipline, and its creation a fabulous leap. A late reflection of this attitude is Michael Mahoney's essay review[2] of the reprint of Neugebauer's *Vorgriechische Mathematik I*, which relegated anything preceding Viète to the realm of "algebraic mode of thought."

In mythology, Athena can be presumed to have sprung fully armed from Zeus's head. In the evolution of mathematics, such miracles are dubious, and the following pages try to dig below the surface, drawing on the much larger amount of sources that are now available. But in order to see what should be looked for, it may be useful to compare an algebra which doubtlessly precedes the supposed leap (al-Khwārizmī's early ninth-century algebra, the earliest extant *treatise* dealing with the topic) to a text which just as obviously postdates it (Leonhard Euler's *Analysis infinitorum*).

First al-Khwārizmī, as Latin Europe encountered him, that is, as translated by Gerard of Cremona; a characteristic passage shows the style:[3]

> Divide ten in two sections and divide one of the two parts by the other, and four result.

1 Ludwig Wittgenstein, *Philosophische Untersuchungen – Philosophical Investigations* (New York: Macmillan, 1953), §§ 66–67.

2 Michael S. Mahoney, "Die Anfänge der algebraischen Denkweise im 17. Jahrhundert," *Rete* 1 (1971): 15–31.

3 Barnabas Hughes, ed., "Gerard of Cremona's Translation of al-Khwārizmī's Al-Jabr," *Mediaeval Studies* 48 (1986): 248; my translation, as all English translations below.

> Whose rule is that you posit one of the two sections to be a *thing* and the other ten less a *thing*. Then divide ten less a *thing* by a *thing* so that four result. However, now you have known that when you multiply what comes out of a division by the same by which you divided, it will give back your amount which you divided. But what results from the division in this question was four, and that by which was divided was the *thing*. Multiply therefore four in a *thing*, and they will be four *things*. Thus four *things* are made equal to the amount which you divided, which is ten less a *thing*. Restore thus ten by a *thing*, and add the same to the four [*things*]. It will hence be that ten are made equal to five *things*. Now you have thus reduced this question to one of the six chapters, which is that roots are made equal to number. [...]

Then two passages from Euler's *analysis infinitorum*, also showing his style of argument when taken together. First vol. I §139:[4]

> Let now in the formulas of § 133 n be an *infinitely small* number, or $n = \frac{1}{i}$, i being an *infinitely large* number. Then
>
> $$\cos nz = \cos \tfrac{z}{i} = 1 \quad \text{and} \quad \sin nz = \sin \tfrac{z}{i} = \tfrac{z}{i}$$
>
> The sine of a vanishing arc $\frac{z}{i}$ is indeed equal to the arc itself, whereas the cosine = 1. Once that is posited one has
>
> $$1 = \frac{(\cos z + \sqrt{-1}\sin z)^{\frac{1}{i}} + (\cos z - \sqrt{-1}\sin z)^{\frac{1}{i}}}{2}$$
>
> and
>
> $$\frac{z}{i} = \frac{(\cos z + \sqrt{-1}\sin z)^{\frac{1}{i}} + (\cos z - \sqrt{-1}\sin z)^{\frac{1}{i}}}{2}$$
>
> [...]

Next the "general equation" for lines of the second order from vol. II §101.[5]

$$yy + \frac{\varepsilon + \gamma}{\zeta}\, y + \frac{\delta xx + \beta x + \alpha}{\zeta} = 0$$

There is no need to go into the mathematical substance of these excerpts. Even a superficial glance will reveal that there are decisive differences and little conspicuous kinship.

1. Al-Khwārizmī's arguments are rhetorical, that is, framed within the syntax of natural language; Euler calculates within the syntax of algebraic symbolism.

4 Leonhard Euler, *Introductio in analysin infinitorum*, vol. 1 (Lausanne: Bousquet, 1748), 104–105.
5 Ibid., vol. 2, 48.

2. Al-Khwārizmī operates with a single unknown named by a word, Euler's range of unknowns has no pre-determined limits, and his unknowns and operators (his "algebraic lexicon") are represented by non-linguistic signs.
3. Al-Khwārizmī's rules and problems deal with the unknown and its second power, Euler again knows no limits.
4. Al-Khwārizmī's coefficients are numerically fixed, those of Euler may have undetermined values.
5. When al-Khwārizmī wants to operate on a composite expression, he needs to take roundabout ways; Euler has the parenthesis.

The parenthesis[6] mostly goes unmentioned when the characteristics of the New Algebra are discussed, and the distinction between symbolic syntax and symbolic lexicon rarely is taken care of. For the rest, the list itself is familiar.

Actually, there is one more fundamental difference: Euler deals with the infinite – here the infinitely great and the infinitely small, elsewhere also with infinite sums. The New Algebra, however, did not do so in general, this only characterized its offspring, "infinitesimal calculus." Moreover, the general parenthesis emerged within infinitesimal calculus, and was only broadly adopted as an algebraic tool after Euler's time.

Euler is thus an approximate representative of the full-fledged New Algebra but does not provide an exact limit *ante quem*. As to *post quem*, we should also be aware that the algebra of the European Middle Ages (and thus ultimately the New Algebra) did not descend from the translations of al-Khwārizmī, nor from the *Liber abbaci*.[7] With minimal lateral input from these, it had sprung from the algebra that was borrowed by abbacus masters in the early 14th century. Ultimately it was based on Arabic algebra, in a tradition which I have characterized as "diluted al-Karajī," but apparently via an earlier Romance-speaking algebra tradition based in the Ibero-Provençal area, which has left no further surviving traces. Since this background is of no importance for the present enquiry, I shall only refer to my earlier writings on the topic.[8]

The earliest abbacus algebra – well represented by that of Jacopo da Firenze, in all probability written in 1307[9] – corresponds well to the characteristics ascribed above to

6 In ordinary text as well as mathematics, a *parenthesis* is that which is enclosed (for instance) in a pair of brackets, not these brackets themselves.

7 This is thoroughly argued in Jens Høyrup, *The World of the Abbaco* (Basel: Birkhäuser, 2024).

8 See Jens Høyrup, "A Diluted al-Karajī in Abbacus Mathematics," in *Actes du 10ième Colloque Maghrébin sur l'Histoire des Mathématiques Arabes (Tunis, 29-31 mai 2010)* (Tunis: Publications de l'Association Tunisienne des Sciences Mathématiques, 2011), 187–197; Jens Høyrup, *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture* (Basel: Birkhäuser, 2007), 153–169; Høyrup, *The World of the Abbaco*, 198–230.

9 Even *if* the algebra chapter in the Vatican manuscript of Jacopo's *Tractatus algorismi* should not come from Jacopo's hand, it can be seen to antedate 1328; and even if it may not be the very first

al-Khwārizmī. It is rhetorical, and makes use of no standard abbreviations that might (or might not) serve as algebraic symbols; it gives rules and illustrates these by means of solved problems; it operates with a single unknown and its powers; its coefficients are numerically fixed; and it has no parenthesis function. The only difference is that its range of powers goes on until the fourth – the *cosa* ('thing'), the *censo* (from Latin *census*, translating Arabic *māl*), the *cubo* and the *censo di censo*. The latter term is meant as a product, *censo*×*censo*, and not as what we might understand as a function, *censo*(*censo*). It gives rules for solving three-member equation types ('cases') involving these powers that can be solved by root extraction or be reduced to the second degree. In Paolo Gherardi's *Libro di ragioni* from 1328 we also find rules provided with examples for solving irreducible cases – obviously wrong, but since they involve radicals and since approximation was not considered that was not immediately conspicuous.

Initial transformation within abbacus algebra

The abbacus school can be traced back to *ca* 1260 – *not* to Fibonacci, as regularly claimed. It taught artisans' and merchants' sons for 1½–2 years, mostly 11 to 12 years old. At that age they had already learned to read and write, the abbacus school taught them how to calculate with Hindu-Arabic numerals on paper (*not* using any kind of abacus) in commercial arithmetic: the rule of three, metrological conversions and shortcuts; simple and composite interest; alloying; perhaps the simple false position; and some rudiments of practical geometry, in particular simple area calculations. Neither the double false position nor algebra were taught. All this follows from two surviving expositions of the curriculum, on the whole agreeing except for minor differences in the order of topics.[10]

However, from the early 14th century onward, such matters entered many of the so-called "abbacus books."[11] Why, if they did not serve in school?

Abbacus master, the teachers of the school, were members of a craft, and as such they were in competition – in small towns, where abbacus schools were organized by the municipality, for positions at these, and in large cities like Venice and Florence, where independent schools could thrive, competition for paying students. For both purposes, formal or informally arranged contests served to display the abilities of the competitors, and these contests had to concern what not every bungler in the trade or any average student

abbacus algebra, it represents the type well.

10 One, from the first half of the 15th century, is a proper description of how teaching was in the "Pisa way"; the other describes the teaching duties of an assistant in Venice in 1519. Details in Høyrup, *The World of the Abbaco*, 6–7.

11 "Abbacus books" is a vague term, referring to all kinds of writings somehow connected to the abbacus school: teachers' books or books for self-study; more or less orderly problem collections; writings by mathematical dilettanti once trained in an abbacus school.

would master. Here algebra came to serve, and several writings show that it was indeed a prestige topic.[12]

This explains that false solutions could flourish – neither municipal councils nor the fathers of hoped-for students would look through them, and only with bad luck would one encounter a competitor who could. It also provided the drive for genuine development, however (more on this below). But let us first look at our chosen basic parameters.

Within a single generation, some writers began using letter abbreviations or non-letter signs for operations (plus, minus, root extraction) and for the unknown and their powers – first for instance ρ or *c* for the *cosa*, later for instance *ce* or □ for the *censo*. In some cases these functioned as genuine *symbols*, in others not.

Since this distinction is not standard, an explanation may be needed. At first we may look at a passage from Biagio "il vecchio"[13]

> Somebody makes two travels. In the first travel he earns 8, in the second he loses at the same rate as he has earned in the first. And then he finds to have 12 *fiorini*. It is asked with how much he set out. Posit that he set out with one *cosa*, at the first travel, and he earned 8, he will thus have a *cosa* and 8. And with this he left the first travel and goes to the second, at which it is said that he loses at the same rate as he has earned. You will say thus: at the first travel he made, from one *cosa*, a *cosa* and 8. And if at the second travel he loses at the same rate, he must do the opposite, that is that you will say, if from one *cosa* he earned 8, he earned $\frac{8}{1\rho}$ of a *cosa*, that is, $\frac{8}{1\rho}$ of his capital, where you will take $\frac{8}{1\rho}$ of a *cosa* and 8, which are 8 *cose* and 84 divided in a *thing*, that is, this fraction $\frac{8\rho\ 64}{1\rho}$.[14] And this is that which he find to have done in the second travel, and we said that he finds 12.[15] $\frac{8\rho\ 64}{1\rho}$ are thus equal to 12. And in

12 Thus Vatican, Ottobon. lat. 3307 (henceforth Ottoboniano *Praticha*) from *ca* 1458, which on fol. 303^r introduces algebra with the words "all that which has been said until this point would be in vain without the present." Luca Pacioli, *Summa de arithmetica, geometria, proportioni et proportionalita* (Venezia: Paganino de' Paganini, 1494), fol. 144^r, "having with God's assistance come to the much desired place: that is, to the mother of all the cases by common people called 'the rule of the thing' or 'the major art', that is, theoretical practice, also called *algebra et almucabala* in Arabic language."

13 Licia Pieraccini, ed., *M° Biagio, Chasi exenplari alla regola dell'algibra nella trascelta a cura di M° Benedetto dal Codice L. VII. 2Q della Biblioteca Comunale di Siena* (Siena: Servizio Editoriale dell'Università di Siena, 1983), 110*f*. This "old" Biagio died before ca 1340. We know an extract from his *Praticha* from Benedetto da Firenze's *Trattato di praticha d'arismetricha*, written in 1463 (this extract fol. 403^{r-v}). Since Benedetto's own notation differs somewhat from what he uses when copying Biagio (and others), we can probably consider him faithful to the original.

14 Addition is indicated by juxtaposition, in agreement with the understanding of addition as aggregation of the constituents.

15 This is actually wrong, it is the loss at the second travel and not what remains; since the rate of loss is 50%, the two coincide, and a numerical check would not reveal the mistake.

order not to have fractions, multiply both sides by 1 *cosa*, you will have 12 *cose* to be equal to 8 *cose* and 64, where you shall confront the sides, removing from both sides 8 *cose*, and we shall have that 4 *cose* are equal to 6, where the *cosa* is worth 16. And we made the position that he set out with a *cosa*, he thus set out with 16.

As we see, Biagio uses the sign ρ within "formal fractions" and operates on them according to the same rules as for ordinary fractions. Others (and sometimes also Biagio) do the same with the *cosa* written in full; that is, they use the *symbolic syntax* without any "symbolic lexicon."[16]

In contrast we may look at Dardi of Pisa, writing in 1344 in a Venetian linguistic environment (where *censo* becomes *çenso*).[17] He uses *c* for *cosa* and *ç* for *çenso* (for ease of distinction I shall write Ç), but does so in a way that *excludes* symbolic operation. When providing them with coefficients he uses a fraction-like notation – instead of 30*c* Dardi mostly writes $\frac{30}{c}$. The underlying idea is that the denominator of a fraction is understood not as a divisor but as a denomination, that is, as an ordinal. Similarly, indeed, in other abbacus texts, for instance the third of three men can be referred to as "the $\frac{1}{3}$," while $\frac{1}{grana}\frac{1}{2}$ may stand for "1 *grana* and ½ [of a *grana*]." This notation, not being *meant* to render a calculation, cannot serve for that purpose; that is, it is *not* symbolic, merely an abbreviation – as made even more obvious by the mixed writing of "1 *censo of censo*" as "$\frac{1}{Ç}$ *de* Ç." Apart from use in manuscripts of Dardi's work I have not noticed this notation elsewhere in Italian manuscripts, but it must have survived – it turns up repeatedly in the wildly eclectic "German algebra" from 1481 (called thus

16 Formal fractions written with full words are used in the *Alcibra amuchabile*, a composite Florentine manuscript which on the faith of indirectly connected watermarks is conventionally dated to *ca* 1365 (also by myself on several occasions) but may be as early as 1335. It explains how to add

$$\frac{100}{\text{per una } cosa} \text{ and } \frac{100}{\text{per una } cosa \text{ e più } 5}$$

by means of the parallel $^{24}/_{4} + {}^{24}/_{6}$, using the rules (syntax) according to which position in fractions determines the roles of and relations between the numbers, just as ordinary syntax determines the relations to ascribe the roles of subject, object and verb (etc.) of words in a sentence, and prescribes rules for performing the operation (here addition), just as ordinary syntax tells us (e.g.) how to change a sentence from the active to the passive voice. The manuscript is transcribed in Annalisa Simi, ed., *Anonimo (sec. XIV), Trattato dell'alcibra amuchabile dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze* (Siena: Servizio Editoriale dell'Università di Siena, 1994), 41.

17 Raffaella Franci and Marisa Pancanti, eds., *Anonimo (sec. XIV), Il trattato d'algibra dal manoscritto Fond. Prin. II. V. 152 della Biblioteca Nazionale di Firenze* (Siena: Servizio Editoriale dell'Università di Siena, 1988), contains an edition of a Tuscan manuscript. Since the abbreviations and the full words do not agree in this Tuscan text, I use the Vatican manuscript Chigi M.VIII.170, where they do. See Høyrup, *The World of the Abbaco*, 230.

because it *is* in German).[18]

Even though signs that *might* serve as symbols were much more common, there was probably little understanding of this distinction at the time. In any case, in the *Divina proportione* Pacioli explains that sundry professionals, among whom the *mathematici per algebra*, use specific *caratheri e abreviature* "in order to avoid prolixity in writing and also of reading" – no symbolic use is hinted at.[19] And indeed, even signs that *might* have served as symbols were mostly used as abbreviations only.

Even though formal fractions are regularly used within the running text, they appear more often as marginal annotations. More complex symbolic calculations are always kept there. We may look at an example from the Ottoboniano *Praticha*, fol. 331^{v}. It deals with a problem which is first expressed in words (as all problems are), but which we may render

$$\frac{100}{1\rho} + \frac{100}{1\rho+7} = 40$$

This is a problem type which since Biagio's time had called for the use of formal fractions. In the margin we actually find the two formal fractions

$$\frac{100}{1\rho} \text{ and } \frac{100}{1\rho+7}$$

and then this scheme

$$\begin{array}{llll} 100\rho & & & \\ \underline{100\rho} & \underline{700\quad\;} & & \\ \underline{200\rho} & \underline{700} & & \\ \;\;1\sigma & \underline{7\rho\qquad\qquad} & & 40 \\ & & & \\ 200\rho & 700 & \text{——} & 40\sigma\ \langle 280\rho\rangle \end{array}$$

corresponding to the calculation

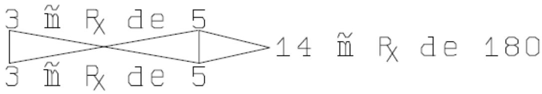
$$\frac{100}{1\rho} + \frac{100}{1\rho+7} = \frac{100\rho+100\cdot(\rho+7)}{(1\rho)\cdot(1\rho+7)} = \frac{100\rho+(100\rho+700)}{(1\rho)\cdot(1\rho+7) = 40} = 40$$

Already Dardi had used schemes to indicate the cross-multiplication of binomials (in his case only arithmetical binomials, but others soon applied them to algebraic binomials, see presently) – for instance,[20] for $(3-\sqrt{5})\cdot(3-\sqrt{5})$

18 Kurt Vogel, ed., *Die erste deutsche Algebra aus dem Jahre 1481* (München: Verlag der Bayerischen Akademie der Wissenschaften, 1981), 10.

19 Luca Pacioli, *Divina proportione*, I (Venezia: Paganino de' Paganini, 1509), fol. 3^{v}. The manuscript (Milan, Biblioteca Ambrosiana, MS 170 Sup., fols. XI^{v}–XII^{r}) specifies, listing abbreviations and signs for *radice*, *più*, *meno*, *quadrato* (*cosa* and *censo* are absent), together (*inter alia*) with abbreviations for *linea*, *geometria* and *arithmetica*. Obviously, not all of these could serve in symbolic calculations.

20 Vatican, Chigi M.VIII.170, fol. 6^{r}.



(℞ abbreviates *radice*, 'root', m̂ *meno*, 'less'). For tri- or longer polynomials this would not do, and no later than the outgoing 14th century additions and multiplications of these were made (still in marginal calculations) in schemes that emulated those used for the corresponding operations with Hindu-Arabic numerals.[21]

Until Pacioli's time, there was no agreement between those who used signs representing powers of the unknown or algebraic operations on *which* signs to use – the only exception being ℞; in Pacioli's words, "as many heads, so many opinions."[22] Few used them *systematically* as abbreviations, and almost none as symbols.

Several unknowns were used by a few, but so few that we cannot establish the connections between them. All those who operate with two unknowns have the *cosa* as their first unknown. In first-degree problems, an entity occurring in the problem – for instance, a price – may be taken as a second unknown. Around 1380, Antonio de' Mazzinghi solved a number of difficult second-degree problems by means of a second unknown called *quantità*. This happens in his *Fioretti*. It is copied conscientiously by Benedetto da Firenze in his *Trattato di Praticha*, which allows us to see that these "small flowers" were a work in progress; here we can follow how the idea of using a second unknown (thus certainly original) develops stepwise. Others must have used the treatise, but before Pacioli we do not know them.

Benedetto himself created a system allowing him to solve first-degree problems by means of symbolic algebra with many unknowns in 1463.[23] Nobody seems to have noticed, and his technique was forgotten.

Pacioli shows us another development. He does so clearly in the *Summa* (just to show how it is done, he explains), but already though confusedly in a large manuscript from 1478–1480 known after the dedication as *Suis carissimis disciplis* (Vatican, Vat.Lat. 3129). It is thus clear that the technique was not invented by Pacioli (it was also used by Nicholas Chuquet in 1484). This technique still uses *quantità* as a second unknown – but then eliminates it, allowing recycling of the name. So, Pacioli and those who inspired him knew to solve first-degree problems with at least three unknowns.

Already Jacopo had made use of four powers of the unknown. That number was soon to grow, and in parallel to the way higher powers were expressed in Arabic algebra (for

21 Thus, frequently, in a *Tratato sopra l'arte della arismeticha* (Florence, BNC, Fondo princ. II.V.152) from the outgoing 14^{th} century, which for binomials uses the same scheme as Dardi. The extensive algebra part of this treatise is transcribed in Franci and Pancanti, *Il trattato d'algibra*.

22 Pacioli, *Summa*, fol. 67^{v}.

23 See Høyrup, *The World of the Abbaco*, 315–325.

instance by al-Karajī), higher powers were initially expressed as products – beyond the *censo di censo* known to Jacopo the *cubo di censo* (or *censo di cubo*), the *cubo di cubo* (or *censo di censo di censo*), and so forth. In the outgoing 14th century, certain writers began transforming this system, and to use sometimes composition based on embedding instead; the above-mentioned Florentine *Tratato* thus uses *censo di cubo* for *censo*(*cubo*), that is, the sixth power, but *cubo di censo* for *cubo*×*censo*, that is, the fifth power.[24] In the late 15th century, embedding had taken over completely, with the consequence that prime powers had to get independent names – the fifth power becoming the *primo relato*, the seventh the *secondo relato*, etc.

Abstract coefficients were only invented by Viète, so there is no reason to discuss them at this point. One "special-purpose parenthesis" was invented, however, first apparently by Dardi. When he wants to indicate that a root is to be taken of a binomial he uses the phrase *R̷ de zonto*, "root of, joined, ..." – for instance,[25] "℞ *de zonto cô* ¼ ℞ *de* 12" standing for

$$\sqrt{1/4+\sqrt{12}}$$

Later writers until Cardano would mostly speak of a "universal" or "bound" root with the same meaning. In principle the notation is ambiguous – how long is the expression meant to serve as radicand? However, only in extremely rare cases is more than a binomial meant.

Antonio goes further in his *Fioretti*. He uses Ⓡ or an encircled fully written *radice* for the universal root. When dealing with an expression corresponding to our

$$\sqrt{cosa+\sqrt{350\text{-}1\ censo}}$$

he refers to the "outer brackets" as "general root," "general root of one *cosa* plus <u>root</u> of 350 less one *censo*" (underlining represents encircling). This terminology was taken over (but misunderstood) by his student Giovanni di Bartolo, after which we have no more traces of it (nor of calculations where it would be needed). Cardano, when eventually encountering the need in the *Ars magna* (fol. 14^{r}), had to reinvent.

The numerator as well as the denominator of formal fractions are also, if composite, special-purpose parentheses, as is the fraction as a whole.

At a pinch, we may also consider the notation for higher powers by embedding (e.g., *censo*(*cubo*)) as a parenthesis; however, the only "argument" (to use the later language of functions) that can be embedded is a power – we never find the *censo* taken of a composite expression, for instance *censo*(5+℞5); this would have asked for a transfer of the notion of "universal" from the domain of roots, but nobody seems ever to have been tempted to

24 The catalyst for this inconsistent system was probably the quest for a way to express roots beyond the square root of the square root.

25 Vatican, Chigi M.VIII.170, fol. 9^{v}.

make that step.

Already al-Khwārizmī sometimes needed to operate on a composite algebraic expression. Then he would interrupt the flow and make a separate calculation; we may see this as a substitute for a parenthesis. In abbacus algebra, such separate calculations often made use of schemes for adding or multiplying polynomials emulating those used in the arithmetic of Hindu-Arabic numerals.

Counting generously, abbacus algebra thus made use of four kinds of special-purpose parentheses; but nobody at the time would see them as instantiations of a single category. There was no general need for that, just as there was no need for a general notion of "function" before the number of these grew beyond powers, sine, tangent, logarithm and exponential.[26]

So, on most of our chosen parameters, abbacus algebra presents us with small steps on the road leading (in retrospect) toward the New Algebra; but they were small, unsystematic, not elements of a search for anything (whatever it might be). To these characteristics we may add an instance of mathematical acuteness. It is a spin-off from the search for solutions to irreducible higher-degree equations, and it turns up twice – first in Dardi.

Dardi offers rules for solving a huge number of seemingly abstruse equation types involving radicals; all are correct, apart from two where he has no name for a fifth and a seventh root. Beyond these he offers rules for solving four irreducible equation types, pointing out explicitly (Vatican, Chigi M.VIII.170, fol. 100^r) that they only work in special cases; they differ in style from the rest, and are almost certainly not his own inventions. He does not explain where they come from, but analysis of the coefficients (made by means of modern letter algebra) reveals the trick: namely a change of variable.[27] Whoever invented these rules must have had a fair grasp of polynomial algebra – a *good* grasp, given that he did not have our symbolism.

The other instance is found in the Florentine *Tratato*. As confirmed by other sources, in the late 14th century a "solution" to equations of the type $K = \alpha t+N$ circulated (N stands for "number," t for *cosa*, K for t^3). As explained in the *Tratato*, the "cube root of 44 with 5 added" is 4 because $4^3 = 44+5\cdot4$.[28] Similarly, the cube root of 65 with 12 added is 5 because $5^3 = 65+12\cdot5$. This is not very interesting. It may be said to offer just a name to the solution; moreover, as the author of the *Tratato* says, it is sometimes but not always possible for given t and N to find a fitting (integer) value of α. Mathematically interesting,

26 The general parenthesis only emerged in the company of the explicit function concept within infinitesimal analysis in the later 17th century; see Jens Høyrup, "How Did the All-Purpose Parenthesis Come About in European Algebra?" *Gaṇita Bhāratī* 45 (2023): 45–75.

27 The principle, expressed in modern notation: The solution to the equation $x^3 = a$ is obviously $x = \sqrt[3]{a}$. Therefore, if $x = y+e$, $y^3+3y^2e+3ye^2+e^3 = a$, and $y = \sqrt[3]{a}-e$. A proper analysis is offered in Høyrup, *The World of the Abbaco*, 262.

28 Franci and Pancanti, *Il trattato d'algibra*, 98.

however, is his explanation (p. 99) of how other equation types – for instance $K+\beta C = N$ – can be transformed into the shape $K = \alpha t+N$. He does not explain in detail, but analysis of the coefficients shows once again that polynomial algebra has been made use of.[29]

Naively we may wonder that Dardi and the anonymous Florentine did not display their ability (or, even more naively, help the collective advance of mathematical knowledge) by explaining their method. Taking into account the social setting of abbacus mathematics we should rather ask, why should they? The abbacus masters, as said, were competing members of a craft, and complicated algebra served in their competitions. Advanced insights within this domain were therefore business secrets. We may remember the Cardano-Tartaglia clash. Cardano, in agreement with the norm system of university learning, published the solution to the cubics; Tartaglia, still living on the conditions of an abbacus master however much he strived to be recognized as a scholar, had needed the solutions to compete.

The lack of uniformity of notations and of systematic use *may* also have something to do with this setting, but it is probably in the main to be explained by the low density of interactions. The use of ρ a a symbol/abbreviation for *cosa* was handed down in a tradition reaching from Biagio to Benedetto, but did not diffuse much laterally – other masters belonged to other school traditions and stuck to their own ways. After all, abbacus communication beyond the single school was mainly carried by manuscripts,[30] and these had limited circulation – only rarely, more than a few copies were made. That was to change when Pacioli's *Summa* was printed in 1494, perhaps in 2000 copies, all of which were sold.[31] After that, the notation for powers found there was adopted by Tartaglia as well as Cardano.

Transmission to German land, and the creation of Coss

Tartaglia and Cardano, however, are outside the main lines of this story. That line, instead, goes through southern German areas.

At least since the early 14th century it had been the habit of long-distance merchants from Nürnberg and other south-German cities to send sons and apprentices (we do not

[29] Analysis in Høyrup, *The World of the Abbaco*, 264.

[30] Mainly, not exclusively. Jacopo da Firenze must have learned his basics at home and then went to Montpellier, apparently to learn more; but his *Tractatus* was then spread in manuscript copies, and his preface in many more. Dardi, from Pisa, seems to have gone to Venice, but again his marvellous treatise went around as manuscripts (four, and a Hebrew translation, have survived).

[31] Alan Sangster, "The Printing of Pacioli's Summa in 1494: How Many Copies Were Printed?" *Accounting Historians Journal* 34 (2007): 125–145. The complete print run must have been sold over the years since the printer made a new edition at his own costs in 1523.

know how many) to Venice in order that they might learn the language and the abbacus.[32]

Only rich merchant houses could afford that, and that provided the occasion for a new profession to arise – the *Rechenmeister*. In 1457, three of these are known to have held school in Nürnberg.[33]

Rechenmeister translates *maestro d'abbaco*, but the style of the teaching as well as the social setting were different. True enough, even the *Rechenmeister* were in competition – the three Nürnberg masters were litigating in court. But one of them published a book, and that symbolizes that competition had moved to a different arena.

Over the next centuries, hundreds of *Rechenbücher* were published, almost invariably claiming on their title page to contain material which is totally new (which is rarely the case). Yet while abbacus algebra thrived within the framework of the abbacus school, German 16th-century algebra (the *coß*) was largely independent. Extremely few *Rechenbücher* present the technique.

The *coß* was an offspring of abbacus algebra, but through particular channels. Around 1460, interest was kindled among (a few) university scholars in a new mathematical technique – new with respect to the mathematics cultivated in their environment. The first phase was eclectic; those who were involved had to draw on whatever they could get hold of, and that can be seen to have been neither coherent nor of high quality. Two participants are known by name: Johannes Regiomontanus, the Vienna astronomer, and Friedrich Amann, who may have gone through Leipzig University but learned (maybe later, when a monk in the St Emmeran monastery in Regensburg) from the Viennese Gmunden tradition.[34] The others are anonymous but are likely also to have come from a university setting.

Among other writings, Regiomontanus studied a manuscript containing Gerard's translation of al-Khwārizmī. The only surviving manuscripts of Robert of Chester's translation of the same work were also written in south German areas.[35] However, influence from either of these in the later *coß* tradition is negligible, restricted to the use of *dragma* as the unit for pure numbers, and widespread awareness that the proper name of the technique is *algebra et almuchabala*.

The names *coß* and *zensus* for the unknown and its square show that northern Italy

32 Bettina Pfotenhauer, *Nürnberg und Venedig im Austausch* (Regensburg: Schnell & Steiner, 2016), 72–76.

33 Eberhard Schröder, ed., *Ulrich Wagner, Das Bamberger Rechenbuch. Facsimile-Druck der Ausgabe von 1483* (Weinheim: VCH Verlagsgesellschaft, 1988), 301.

34 See Armin Gerl, "Fridericus Amann," in *Rechenbücher und mathematische Texte der frühen Neuzeit*, ed. by Rainer Gebhardt (Annaberg-Buchholz: Adam-Ries-Bund, 1999), 1–12; Dana Bennett Durand, *The Vienna-Klosterneuburg Map Corpus of the Fifteenth Century* (Leiden: Brill, 1952), 73–76.

35 See Barnabas B. Hughes, ed., *Robert of Chester's Latin Translation of al-Khwārizmī's Al-jabr* (Wiesbaden: Franz Steiner, 1989), 11–13.

(Venice etc.) and not Tuscany provided the main inspiration – in the north they were written *cossa* and *zenso* in the 15th century.

The details of this eclectic phase do not concern us here.[36] The first extant orderly presentation – and the earliest extant representative of the *coß* – is a Latin algebra[37] contained (together with numerous other mathematical treatises) in the manuscript Dresden, Sächsische Landesbibliothek, C 80. This manuscript was once in the possession of Johannes Widmann, and the Latin algebra is likely to have been the basis for the algebra lectures he offered in Leipzig in 1486. We have no evidence that these lectures had any influence, but the contents of the Latin algebra had.

Actually, the treatise in itself is a conglomerate, but the three components supplement each other without contradicting, showing that this contents already existed as a whole.

The treatise makes systematic use of abbreviations for the first powers of the unknown:

φ	*numerus*
2e	*res*
ʒ	*zensum*
cf	*cubus*
ʒʒ	*zensum zensorum*

Less systematic is the use of a dot «·» for *radix*, "root,"[38] and «··» for "root of root." Cube root, instead, when not written in full, is "℞ *cubica*"; the square root also occasionally appears as ℞.

The Italians, when abbreviating *più* ('more', whence 'plus') and *meno* ('less', whence 'minus'), had used letter abbreviations. The present manuscript instead uses + and –. Like the transformed dot, even these signs, mostly serving as symbols, were also taken over by others and ultimately by modern mathematics.

Where they can serve, formal fractions are used. No other instances of symbolic calculations are present.

The next, and decisive, step toward the formation of the *coß* was undertaken by Andreas Alexander, one of the first specialist lecturers on mathematical topics (meaning

36 They are described in Høyrup, *The World of the Abbaco*, 389.

37 Edited in Hermann Emil Wappler, "Zur Geschichte der deutschen Algebra im 15. Jahrhundert," in *Gymnasium zu Zwickau. Jahresbericht über das Schuljahr von Ostern 1886 bis Ostern 1887* (Zwickau: R. Zückler, 1887), 1–32.

38 This dot, when written on uneven paper, was often written with a down- and an upstroke, as ✓. This gave rise to the modern root sign √. Once this transformation was made, for instance in Christoph Rudolff, *Behend unnd hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt werden* (Straßburg: Johann Knoblauch, 1525), and Michael Stifel, *Arithmetica integra* (Nürnberg: Johan Petreius, 1544), a dot after the root sign could be used to indicate a universal or bound root.

he was not allowed to go on with other, more lucrative topics; certainly no promotion). Before 1504 he wrote a Latin algebra manuscript. One version (Leipzig, Hs. 1696) was discovered by Menso Folkerts. Beyond that, we have a German commentary (*Initium Algebrae*, "Algebra's Introduction") surviving in four copies, which explains a somewhat different, later version of the Latin treatise.[39] The existence of at least two different manuscripts, and of a number of copies of the German commentary, explains that Alexander's work could become influential.

Alexander extends the sequence of names of powers and appurtenant signs

φ	1.	dragma or numerus
ꝶ	2.	res
ʒ	3.	zensus
ꝗ	4.	cubus
ʒʒ	5.	census de censo
ß	6.	sursolidum
ʒꝗ	7.	censicubus
bß	8.	bissursolidum
ʒʒʒ	9.	census censu de censu
ꝗꝗ	10.	cubus de cubo

As we observe, he also numbers them – unfortunately counting *number* as the first, which makes the formulation of rules for their product cumbersome; Pacioli had made the same unlucky choice a few years earlier, but whether there is any influence is undecidable. In any case, in the later version which is commented upon in *Initium Algebrae* he drops this infelicitous numbering. Formal fractions are not only used but also explained; operations on polynomials are made within schemes emulating those used for calculation with Hindu-Arabic numerals. In *Initium Algebrae* it is even shown how to multiply two formal fractions whose numerators themselves are formal fractions.[40] Calculations with quadratic and cubic surds are explained with reference to geometric diagrams (the latter in perspective drawing).

The Latin algebra from C 80 had offered 24 rules – the 22 rules for reducible cases that had been current in abbacus algebra (20 of them since Jacopo), and two (equally reduc-

39 This follows from ongoing work by Martin Hellmann, who also corrected the interpretation of the title (Martin Hellmann, "Von Pythagoras zum Liber de cosa: Bemerkungen von Nicolaus Matz über die Geschichte der Rechenkunst," in *Bewahren und Erforschen: Beiträge aus der Nicolaus-Matz-Bibliothek (Kirchenbibliothek) Michelstadt*, edited by Wolfgang Schmitz (Michelstadt: Stadt Michelstadt, 2003), 88). Maximilian Curtze, ed., *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance* (Leipzig: Teubner, 1902), 435–600, is an edition of what he named *Initius Algebras*. In its prologue (fol. 41^{v}) and explicit (fol. 127^{v}), even the Latin text refers to the work as that of *Algebras arabus*.

40 Curtze, *Urkunden*, 517.

ible) involving roots; at the same time it had explained how to reduce higher-order cases. Alexander cuts down their number to eight, leaving out those that can be easily reduced.

The first to bring German algebra into print (now German also in language) was Heinrich Schreyber alias Grammateus, in *Ayn new kunstlich Buech, welches gar gewiß und behend lernet nach der gemainen regel Detre, welschen Practic, regel falsi unn etlichen regeln Cosse* from 1521. Quite unusually he did so as part of a *Rechenbuch* (obviously, norms for what belonged within the genre had not yet crystallized). He identified powers not by means of signs but by numbers (coinciding with our exponents). Since no repercussions of his work can be traced there is no reason to discuss it further.

The book that instead came to define *coß* was Christoph Rudolff's *Behend unnd hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt werden* from 1525 (henceforth Rudolff's *Coss*). Beyond its being printed and its lucid style, one reason was probably that Rudolff offered a large number of problems illustrating the use of the eight rules.

Rudolff owes much to Alexander. He presents the same eight cases (making fun of those who make empty generalizations to 24, offering a hundred if wanted); he uses the same names and signs for the powers of the unknown; and his arithmetic for monomials and binomials is similar. He eliminates the geometric demonstrations used by Alexander to support the explanation of the arithmetic of mono- and binomials (as well as those serving in the *Initium Algebrae* for the fundamental algebraic cases).

In a way we might claim that there is nothing new in what was done by Alexander and Rudolff. Nothing they do had not been done already by *some* abbacus writer. We may even observe that we find in the cossic writings nothing similar to the secret explorations of polynomial algebra and its application to irreducible higher-degree equations. There is, however, a fundamental difference: Since the Latin algebra of C 80, cossic algebra had used the same notation (leaving aside Schreyber's vain attempt to introduce a theoretically based alternative), and it was using it systematically – so systematically that it is often difficult to find out which verbal names for the powers were intended.

Most of what Rudolff offers (though not his abundance of examples) goes back to Alexander. Not everything, however, does, Rudolff also adds. In particular – of importance for the future of algebra – he adds the *regula quantitatis* and uses it extensively.

The *regula quantitatis* was not new. It is simply the technique to eliminate and then recycle a second unknown – the *quantitet*, thereby allowing the operation with more than two unknowns. The technique was mentioned above in connection with Pacioli, together with its use by Chuquet in 1484 – thus after the writing of the *Suis carissimis*, which already suffices to show that Chuquet was not Pacioli's source. Inspiration the other way can also be excluded, Pacioli's presentation in *Suis carissimis* being confused.

The "rule" – and now also the name, unknown to Chuquet and Pacioli – are also used by Étienne de la Roche in 1520.[41] Albrecht Heeffer takes this to be an indication that de la Roche was Rudolff's source,[42] but at the same time points out that a marginal note written into the Chuquet manuscript,[43] almost certainly by de la Roche, states that *ceste regle est appelee la Regle de la quantite*, "this rule is called the rule of the quantity," showing that de la Roche has discovered use of a rule he already knows under this name. None the less, striking unlikely similarities between de la Roche's text and that of Rudolff make it next to certain that Rudolff knew de la Roche's book.[44]

De la Roche uses the method much less than Rudolff, however, and his book was not nearly as well diffused. Already for this reason there is little doubt that the after-world came to know the method mainly through Rudolff.

This "completion of the *coss*, indeed in truth a completion without which it would not be worth much more than a trifle"[45] became really important through the transformation it underwent in Michael Stifel's *Arithmetica integra* from 1544 and in particular in his re-edition of Rudolff's *Coss* from 1553.

Stifel's "Whole of arithmetic" consists of three books. Book I may be characterized as a regular though broadly considered arithmetic. Book II is, so to speak, an arithmetical transformation of *Elements* X. Book III (fols. 227^{r}–319^{r}), finally, is an orderly algebra in the *coß*/Rudolff tradition.

Already therefore it was an important book. Few outside German lands at the time were able to read German, so the improvements of *coß* algebra over the abbacus tradition – namely the insight that a symbolic notation is next to worthless if it is not used systematically, and in the same way by everybody – only reached them through Stifel.

Even more important, however, was certainly his transformation of the *regula quantitatis* (as I shall argue). Stifel was the first to propose a system for the use of many unknowns (Benedetto had not explained a system, he just developed the technique, soon to be forgotten). On fol. 251^{v} we find the heading *De secundis radicibus*, "on second(ary) roots." Stifel still uses *res* (written 𝓇) as his primary unknown, but instead of a *quantitet* that can be recycled he proposes an open-ended sequence of secondary unknowns, using the letters of the alphabet: "1*A* (that is, 1*A*𝓇), 1*B* (that is, 1*B*𝓇), 1*C* (that is, 1*C*𝓇), 1*D* etc."; for their second powers he uses 1*A*ʒ etc. For the product of 𝓇 and *A* he suggests 𝓇*A*,

41 Étienne de la Roche, *Larismethique novellement composee* (Lyon: Constantin Fradin, 1520), fols. 42^{r}, 61^{r}.

42 Albrecht Heeffer, "The Rule of Quantity by Chuquet and de la Roche and Its Influence on German Cossic Algebra," in *Pluralité de l'algèbre à la Renaissance*, edited by Sabine Rommevaux, Maryvonne Spiesser, and Maria Rosa Massa Esteve (Paris: Honoré Champion, 2012), 127–147.

43 Paris, BNF, français 1346, fol. 169^{v}.

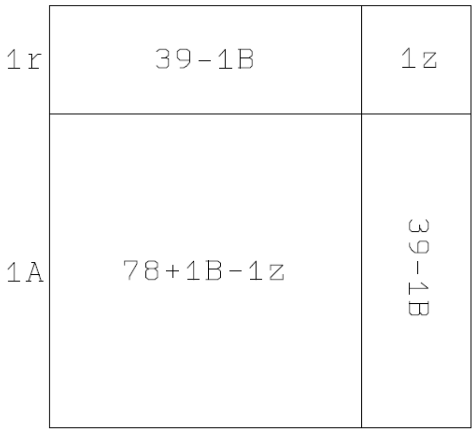
44 Jens Høyrup, *Explorations and False Trails* (Cham: Springer, 2024), 106.

45 Rudolff, *Behend unnd hübsch Rechnung*, fol. I.vi^{v}.

while that of *A* and *B* will be written *AB*.[46]

The first illustrating example asks for the splitting of a given number in parts with given ratio. It is taken from Rudolff, who solves it on fol. 64^{r} by means of a single unknown, as everybody had done since al-Khwārizmī. The second asks for seven numbers (here supposed to be debts owed to Stifel), given the sums of all except the first, except the second, except the third, etc. That type (we may call it "all except each") was traditionally solved without any algebra.[47] Stifel uses his new technique, in a way which would easily have yielded to the traditional *regula quantitatis*.

The third example is a reducible quartic, thus more advanced. Actually, apart from Antonio (and Pacioli borrowing one of his problems), nobody known to us in the abbacus or derived traditions had used several unknowns in non-linear problems. The problem (fol. 254^{v}) asks for two numbers (say, *P* and *Q*) fulfilling the condition



$$P^2 + Q^2 - (P+Q) = 78\,,\; PQ + (P+Q) = 39.$$

Stifel chooses the first number to be 2ℓ, the second to be *A* and their sum to be *B*. He proceeds in a way that owes more to *Elements* II or square-grid geometry than to algebra, using the adjacent diagram.

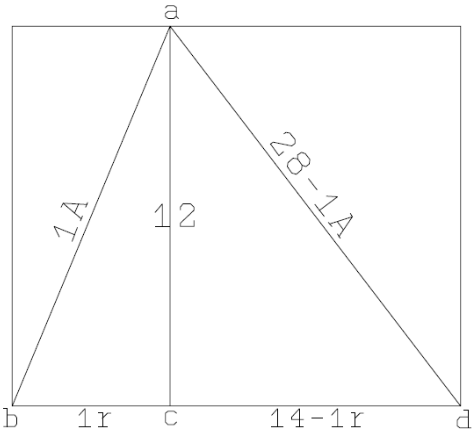
The second condition gives him 2ℓ *A* = 39–1*B*. Thereby he can complete the square, etc., without using algebra. As we see, the geometric interpretation allows Stifel to take over from geometry the habit of naming more than a minimal set of unknowns by letters (a point to which we shall return).

The end of the *Arithmetica integra* is a collection of supplementary problems. Most of those on fols. 292^{r}–301^{r} make use of the new system, and with two exceptions they are all of the first degree (and thus uninformative).

One non-linear problem (fol. 292^{r}) asks for three line segments for which the areas of the rectangles they contain pairwise are given. They are named 2ℓ, *A* and *B*. The solution follows from another square-grid diagram, without any use of algebra or formulation of an equation.

46 The explanation of 1*A* as standing for 1*A*2ℓ shows that *A*, *B*, etc. are thought of as markings. So, all the first powers are 2ℓ, but they are distinguished as ^{A}2ℓ 'the *A*-kind of 2ℓ'), ^{B}2ℓ, etc. When standing to the left, on the other hand, 2ℓ is meant as a factor. This system might easily produce mistakes, and in 1553 Stifel would abandon it.

47 The sum of all given numbers is seven times the sum of the debts minus a single time this sum, that is, six times the sum of the debts.



The final problem (fol. 300^v) is more intricate than the others, also according to Stifel, and meant to illustrate the "immense usefulness of the second roots." It is properly geometric but solved by means of algebra. It deals with a rectangle with sides 12 and 14, subdivided into two rectangles, the sum of whose diagonals is 28. From the Pythagorean theorem follows that ab^2 is 12^2+(12ℓ)^2 = 1ʒ+144, identified with 1*A*ʒ (i.e., $(1A)^2$). ad^2 is found in a similar way to be 340–282ℓ+24ʒ; but it is also $(28–A)^2$ = 784–56*A*+1*A*ʒ. Therefore, 1*A*ʒ = 56*A*+1ʒ–282ℓ–444. But 1*A*ʒ = 1ʒ+144, so *A* can be eliminated. The problem is thus of the second degree, but only in 2ℓ.

In 1553, Rudolff's *Coss* had become so rare that it was no longer to be bought "even if one wanted to pay three or four times" the price, for which reason Stifel was asked to prepare a new edition (as Stifel tells in his preface). This new edition appeared as *Die Coss Christoffs Rudolffs. Die schönen Exemplen der Coss gebessert und gemehrt*. Here, Stifel replaces Rudolff's use of the *Regula quantitatis* with his new system, also using it in a number of problems where Rudolff had used a single unknown.

That, of course, does not tell us much. More interesting is a collection of 24 new sophisticated problems added in the end – appropriate to the different context of the book, where problem solution and not the presentation of theory is central.

Half of the new problems (all of higher degree) make use of several unknowns. One of them appears to be borrowed from Pacioli (*Summa*, fol. 148^v), who had already borrowed it directly or indirectly from Antonio: a question for two numbers fulfilling the conditions $mn = 96$, $m^2+n^2 = 292$. Stifel makes the positions *m*:= 2ℓ+*A*, *n*:= 2ℓ–*A*. Pacioli's parameters are different but his positions the same. The remaining 11 problems appear to be new. Two are solved by means of geometric diagrams similar to those used in the *Arithmetica integra*, the others by means of algebra. All would be very difficult without use of the new technique.

Stifel has now simplified his notation, leaving behind the idea that all unknowns are different kinds of 2ℓ. Instead of *A*ʒ Stifel now writes *AA* (etc.); powers until *k*, ʒ*A*, 2ℓ*AA* and *AAA* appear (fol. 469^r).

The echo was faint, few German writers on algebra believed they had any use for Stifel's system (we shall return to French writers). Caspar Peucer's Latin *Logistice Regulae Arithmeticae, quam Cossam & Algebram quadratam vocant*, the second part of his *Logistice Astronomica Hexacontadῶn et Scrupulorum Sexagesimorum* […], contains a section *De radicibus secundis*, "on second roots" (fol. T.vi^{r}). It refers to Rudolff, Cardano[48] and Stifel. The notation is that of the *Arithmetica integra* ("1 *A*, id est 1 A2ℯ," etc.). The book was meant for the higher Lutheran educational system and had scant influence; its examples are utterly simple.

Mattheus Nefe's *Zwey newe Rechenbuecher* from 1565 does not treat algebra in general (that, as said, falls outside the scope of standard *Rechenbücher*); once, however, Nefe gives a single example of *Regula quantitatis* of type "all except each."[49] In so far it is uninteresting; Nefe, however, has seen that there is no reason to distinguish between primary and secondary unknowns, so his unknowns are simply *A*, *B*, *C* and *D*.

Finally, in his *Algebra*, Christopher Clavius (German at least by birth, though working in Rome and writing in Latin) presents the system from the *Arithmetica integra* faithfully.[50] On pp. 56–70, 15 problems illustrate its use. The first three coincide with Stifel's first three (the first and second changing the parameters but nothing else); the fourth is a very simple second-degree problem.[51] 11 are of traditional recreational type ("give-and-take," "finding of a purse," "purchase of a horse"). No wonder perhaps that nobody seems to have taken notice among the many students of the Jesuit schools who were taught by Clavius's book (including Descartes, as we shall see).

Rounding off, we observe once more that *Rechenmeister* algebra was not closely associated with general *Rechenmeister* mathematics. *Rechenbücher* might present the double false position, even though even that was not of real practical use; but only Schreyber dealt with algebra.

Some *Rechenmeister* may have followed the arts course of a university, but as a rule they had not. Writers on algebra, to the contrary, had fair Latin learning (whether obtained during a regular inscription or not).

Algebra was known as an art serving to solve problems, and *Rechenmeister* algebra did so too. In so far it followed the norm system of *Rechenbücher* – after all, this orientation agrees well with the inherited Italian model.

University mathematics was different.[52] At the lower level it taught Boethius's *Introduction to Arithmetic*, and also algorism, the calculation with Hindu-Arabic numerals includ-

48 Girolamo Cardano, *Practica arithmetice, et mensurandi singularis* (Milano: Bernardini Calusco, 1539), fol. L.viii^{r}, follows Pacioli, referring to the second unknown as *quantitas surda*. In Girolamo Cardano, *Artis magnae sive de regulis algebraicis, liber unus* (Nürnberg: Johan Petreius, 1545), fol. 21^{v}, it has become *secunda quantitas incognita*.

49 Mattheus Nefe, *Arithmetica: Zwey newe Rechenbuecher* (Breslau: Crispin Scharffenberg, 1565), fol. P.iii^{r}.

50 Christopher Clavius, *Algebra* (Roma: Bartolomeo Zanetti, 1608), 72–76.

51 The structure is $x^2+a^2 = 340$, $xa = {}^{6}/_{7}\, x^2$. Clavius may have borrowed it from Jacques Peletier, *L'algebre* (Lyon: Ian de Tournes, 1554), 102.

52 An overview can be found in Jens Høyrup, "Mathematics Education in the European Middle Ages," in *Handbook on the History of Mathematics Education*, ed. by Alexander Karp and Gert Schubring (New York: Springer, 2014), 109–124.

ing sexagesimal fractions. At the higher level, at least the first part of the *Elements* or Witelo's *Perspective communis*, and (probably for the few, but well known to the best Vienna mathematicians) *theory* – that of Jordanus of Nemore, and of the Oxford *calculatores* and continental followers like Albert of Saxony and Nicole Oresme. Lectures on theory were certainly followed by disputations, but from their reflection in collections of *quaestiones* we see that they would concentrate on foundational problems (Oresme's questions on Euclid being in part an exception, since Oresme used the opportunity to produce *more theory*). There was no opportunity to develop a culture of problems.

This aspect of university mathematics influenced even that of the writers on *Rechenmeister* algebra. Firstly, many of them, from Alexander onward, presented the arithmetic of irrationals much more thoroughly than had been done in abbacus algebra.[53] Secondly, the problems presented in *Rechenmeister* algebra are really meant as illustrations, there is little agonistic ostentation; we may even say that Rudolff, when poking fun at the 24 rules, censures them as empty display.

Thirdly, as far as it goes the use of symbols (signs for the unknown and for operations, the use of formal fractions and of schemes for operations on polynomials) is systematic, and very homogeneous. As pointed out by Franz Woepcke in connection with his description of the Maghreb algebraic notation,[54]

> the indispensable condition for characterizing a set of conventional signs as a notation is that they are always used when it is fitting, and always in the same way.

The role of a single book – that of Rudolff – as the one everybody had read and emulates (a "paradigm" in Thomas Kuhn's original sense) of course helped; but only because Rudolff himself had been systematic within that book.

Repercussions of Stifel's system outside the German area

As we have seen, there were almost no repercussions of Stifel's new system within the German area. Elsewhere?

I have only been able to trace four pertinent authors. Two seem to have been uninfluential, a third *may* have provided Viète with some inspiration, the fourth though almost unnoticed was probably quite important.

53 In a way, Benedetto is an exception – but what we find in his *Trattato di praticha* is a copy of an arithmetical interpretation of *Elements* X which he has found ready-made (Høyrup, *The World of the Abbaco*, 330).

54 Franz Woepcke, "Recherches sur l'histoire des sciences mathématiques chez les Orientaux, d'après des traités inédits arabes et persans. Premier article. Notice sur des notations algébriques employées par les Arabes," *Journal Asiatique*, 5e série, 4 (1854): 355.

The first was Jacques Peletier, who follows the *Arithmetica integra* without adding anything conspicuous. We shall say more about him below.

The second is Jean Borrel (Latinized Buteo), who used several unknowns in his *Logistica, quae et arithmética vulgò dicitur* from 1559. Borrel is one of those who rarely cite predecessors except in order to castigate them. Moreover, he often changes the terminology he has inherited in order to make the exposition look more Greek (the *Logistica* of his title is an example). We must therefore rely on indirect arguments in order to determine from where he got his ideas.

On p. 189 he introduces "another way of calculation which *vulgò* is called the Rule of quantity." He states that he will *not* follow the shape given to it by Luca [Pacioli] or Stephanus [de la Roche] or others because that shape is extremely troublesome [*molestissima*]. Since Borrel is not afraid of mentioning Cardano (he does so on p. 188), the unidentified "others" can hardly be anybody but Stifel (or Peucer or Valentin Mennher, on whom imminently, both of whom follow Stifel). Since his unknowns are 1 *A*, 1 *B*, etc., Borrel has simply done as Nefe was to do again a few years later: he gives up the idea that one unknown in primary and the others secondary.

On pp. 189–196, four examples are offered, and on pp. 357*f* a fifth. They are all linear and variations of such classics as "give-and-take" and "finding of a purse." Hardly likely to provoke much interest.

The third is Guillaume Gosselin, to whom we shall also return. Even he censures predecessors (Pacioli, de la Roche, Cardano, Borrel "and others" without specifying in what they err).[55] In any case, he appears to imitate Borrel with a small admixture from Pacioli or Cardano. The topic fills book IV of the work. At first, under the heading "De quantitate absoluta" (fol. 80^{r}), five problems are dealt with in Borrel's notation; next, with heading "De quantitate surda" (fol. 84^{e}), four problems are dealt with as Pacioli would have done, the second unknown being *q* (without recycling, however). All problems are of the first degree, of classical recreational types though formulated in terms of pure numbers and not as dealing with monetary possessions or prices. It is noteworthy that *A* and *B* are used to represent numbers in *arithmetical* argument on fols. 27–33 and 58–62.

Mennher's *Arithmétique seconde* from 1556 did provoke interest – exceptionally long-lasting interest indeed. In 1620, Didier Henrion was to publish *Deux cens questions ingénieuses et récréatives extraictes et tirées des oeuvres mathématiques de Valentin Menher*, in part drawn from the *Arithmétique seconde*, in part from Mennher's spherical trigonometry;[56] at his death in 1637, Descartes' friend and mentor Isaac Beeckman possessed a

55 Guillaume Gosselin, *De arte magna, seu de occulta parte numerorum quae & algebra, & Almucabala vulgo dicitur* (Paris: Egide Beys, 1577), fol. 80^{r}.

56 Half drawn indirectly, however, from the similar collection of 100 problems published by Michiel Coignet (who had probably studied with Mennher) in 1573 in Antwerp. See Ad Meskens, *Practical Mathematics in a Commercial Metropolis* (Dordrecht: Springer, 2013), 59.

copy;[57] and as late as 1666, John Collins points to Mennher together with Viète and Viète's translator Jean-Louis Vaulezard (etc.) as good introductions to algebra.[58]

Mennher was born in Schwaben in 1520.[59] He entered the Fugger firm as an accountant, which brought him to Antwerpen. He settled there and opened a school, and also started mathematical publishing in French (thereby opening the world of *Rechenmeister* mathematics to French readers). According to the preface to his "second arithmetic" he had published a first arithmetic in 1550 – probably part of his *Practicque brifue*, of which I have only been able to see a re-edition of the book-keeping part.

A first part of the *Arithmetique seconde* is a good *Rechenbuch* in German style. The third is a geometry going well beyond traditional abbacus geometries – it contains a Euclidean *proof* of the Pythagorean theorem (fol. S.i^{r}) as well as an Archimedean determination of the ratio between the perimeter and the diameter of a circle (our π). This is not our only evidence that Mennher was a competent geometer: in 1564 he published "practice of spherical triangles, the distances on the globe, clocks, shadows, and other ingenious and new mathematical questions"[60] – adding, as said in the preamble, to what had been done by the very learned Regiomontanus (*viz*, in *De triangulis*) the labour of calculation.

Between the first and the third part we find an algebra. As in the case of the spherical triangles, Mennher is not afraid of identifying sources. In the present case the "style and manner" of the very renowned Christoph Rudolff has been of great help, and Mennher has found him very competent. In consequence, he says, he has not deviated much from him, knowing well that the very renowned Michael Stifel has renewed and augmented him much in the same High German language with several beautiful examples – from which, however, "I have extracted a fair part of the best only, adding other matters needed by merchants."

At first comes a presentation of traditional *coß* with a single unknown, which fits the reference to Rudolff. It is followed by an exposition of the *regle de la quantité ou seconde radix*. It is inspired by Stifel, in agreement with what was announced, but Mennher does not copy. His notation is Stifel's improved version from 1553. Since no existing books apart from that of Stifel treated the topic in depth, Mennher must have been working on it directly.

The first illustrative example (fol. O.i^{v}) is an "all less each" problem with four participants. It is similar to one in Stifel's Rudolff-edition (fol. 312^{r}), but the parameters are not

57 See *Catalogus variorum et insignium librorum clarissimi doctissimique viri D. Isaaci* [...] (Dordrecht: Isaac Andreae, 1637), B.iv^{v}.

58 Philip Beeley and Christoph J. Scriba, eds., *The Correspondence of John Wallis*, vol. II (1660–September 1668) (Oxford; New York: Oxford University Press, 2005), 315.

59 See Meskens, *Practical Mathematics*, 14*f* and *passim*.

60 Valentin Mennher, *Praticque des triangles sphériques, des distances sur les globes, et autres ingenieuses et nouvelles questions mathematique* (Antwerpen: Gilles Coppenius, 1564).

the same; even the choice of the unknowns is different, for which reason Mennher cannot follow Stifel's procedure. The third example (fol. O.ii^{r}) is similar to Stifel's two-number problem from the *Arithmetica integra* (fol. 254^{v}), but again the parameters are different; the argument is similar (including the use of a convenient though somewhat superfluous third unknown). Other two-number problems follow patterns we know from Stifel while being solved differently – sometimes making a more convenient choice of the algebraic unknowns, sometimes by using a diagram instead of algebra; still other examples have no analogue in Stifel. Mennher was to present the *regle de la quantité* once again in 1565,[61] by then expanding the treatment though not much. However, the book possessed by Beeckman and referred to by Collins appears to be the *Arithmetique seconde*.

Mennher thus produced no striking innovations; but he, and apparently nobody else, took care that the mature and full version of Stifel's technique from 1553 could reach French readers. Actually he was widely plagiarized also for other matters, including in Flemish, thereby diffusing more basic *Rechenmeister* mathematics in the Netherlands and France, but that is irrelevant for the present purpose.

France before Viète

The New Algebra emerged in France. That is the reason that Mennher's transmission of *coß* and of Stifel's system should be taken note of. But there was also interest in algebra in 16th-century France proper.

The earliest manifestation of this interest was de la Roche, whose book, as we have seen, probably provided some information to Rudolff. He may also have had peripheral influence on Mennher – the latter speaks of *racine universelle* as well as *racine liee*. Both terms are used by de la Roche. But Mennher uses *racine liee* about the sum of two roots – that is, *if* borrowing the terms from de la Roche he reinterprets them so as to fit a distinction made in *coß* algebra.

De la Roche's influence in later properly French algebra seems to have been equally modest. The lack of system in his notations – in absolute contrast to what we find in the *coß* – will hardly have helped. The next evidence of algebraic interest is a re-edition in 1551 in Paris of Johann Scheubel's algebraic introduction to his edition of *Elements* I–VI.[62] Since it was republished in 1552, there must have been an interested audience – but Scheubel's rather idiosyncratic notation left no visible traces, so even *his* influence must have been modest.

[61] Valentin Mennher, *Praticque pour brievement apprendre à ciffrer, et tenir livre de compte, avec la regle de coss, et geometrie* (Antwerpen: Aegidius Diest, 1565), fols. F.fi^{r}–G.gii^{v}.

[62] Johann Scheubel, *Algebrae compendiosa facilísque descriptio, qua depromuntur magna arithmetices miracula* (Paris: Guillaume Cavellat, 1551).

In 1554, Peletier published *L'algebre* in Lyons. On the first pages he lists those writers on the topic whom he knows about – also those whose works he has not seen, including Rudolff; noteworthy is that he does not know about de la Roche.[63]

His own work builds on Stifel's *Arithmetica integra*, of which he makes no secret (including the arithmetical treatment of the matters of *Elements* X); from Stifel he has taken over the insight that a notation has to be used systematically, only making some purely graphical changes of the signs; for the notation of "second roots" he follows Stifel precisely.[64]

Peletier thus introduced the style and insights of German *coß* to French readers; but since nobody followed him he can hardly be said to have *brought them* to France.

Borrel's *Logistica* from 1559, also published in Lyons, was already mentioned. Borrel's principal aim is to assimilate arithmetic and algebra to some Greek-sounding model. Like Peletier (and Scheubel) he accepts that a notation has to be systematic. But he replaces the already current + and – with *P* and *M*; for the powers he uses the Florentine ρ (which he is likely to have known from de la Roche) for the first power, ◇ for the second and ❐ for the third. All in all the book is a protracted primer, and it left no obvious traces.

Petrus Ramus's utterly elementary *Algebra* was published anonymously in Paris in 1560. Ramus uses + and –, "seized upon from common usage." Since he pretends implicitly in the *Scholarum mathematicarum libri unus et triginta* from 1569 not to know Stifel we may safely assume (given what else he pretends not to know there) that he has drawn on him (possibly through Peletier, but even then he would know about Stifel whom Peletier cites repeatedly). Ramus invents new signs (simple and composite letter symbols) for the powers, and employs the usual schemes for the arithmetic of polynomials. Even this book got no following.

Guillaume Gosselin's *De Arte magna, seu de occulta parte numerorum, quae & Algebra, & Almucabala vulgo dicitur* was published in Paris in 1577; it presents us with another abortive fresh start. After the appearance of Xylander's Latin translation of Diophantos[65] it takes over much of Xylander's terminology and notation but modifies it as actually required (Diophantos had produced higher powers as products, Gosselin as everybody since a century by embedding); whereas Xylander had used + and –, Gosselin writes *P* and *M*.

In 1578, Gosselin published an abbreviated French translation of the first and second part of Tartaglia's *General trattato di numeri, et misure* (the second part of which deals with

63 The Latin version from 1560 does mention de la Roche in its preface.

64 Omitting however Stifel's first example because it is easily solved by means of a single unknown, and replacing it with the problem later used by Clavius, see note 51. We may assume that it was created by Peletier, since two other problems are correctly said on p. 107 to be borrowed from Cardano (namely, from Cardano, *Practica arithmetice*, fols. L.vii^{r} and HH.vi^{r}).

65 Wilhelm Xylander, ed. and trans., *Diophanti Alexandrini Rerum arithmeticarum libri sex* (Basel: Eusebius Episcopius, 1575).

algebra and its foundations)[66] with added commentaries from his own hand and using his own algebraic notation.[67]

There was thus a continuous interest in algebra in the French area ; to the actual writers about the topic we may add Pierre Forcadel, who in 1556 announced on the title page that his *Arithmeticque* would allow to "easily obtain knowledge of algebra" without actually dealing with that subject.[68] But there was very little continuity, and thus *no algebraic tradition* from which the New Algebra could emerge.

Schematically – where ≢ indicates lack of connection:

Étienne de la Roche (1520) ≢ [Scheubel (1551)] ≢
Jacques Peletier (1554, Latin 1560) ≢ Jean Borrel/Buteo (1559) ≢
Pierre de la Ramée (1560) ≢ Guillaume Gosselin (1577)

A new cultural setting

That emergence, instead, owed much to two aspects of a new cultural setting.

One aspect is the advent of "Humanist mathematics." Original Humanism (that of Petrarca, Boccaccio, etc.) could not have cared less about mathematics. In spite of its professed admiration for Greek Antiquity, it read Latin, and that gave access to nothing beyond Boethius (Campanus's Latin translation of the *Elements* belonged to the university tradition). After 1450, that began to change.

From its beginning, Humanist thought had been interested in "utility," understood as civic utility – and that again understood as that which could embellish the state or serve its diplomatic or military strength. During the later 15th century it became clear that this service had to include (applied) mathematics – in great drainage projects and other engineering undertakings, in fortification, and (in particular in the Atlantic states) in navigation. That created the new social role of a "court mathematician"[69] – and that established the contact between Humanism and mathematics, and made mathematics a Humanistically legitimate interest.

Legitimacy takes its time to be distilled into results. The first Humanist translations of Greek mathematics were published only from around 1500 onward:

– Giorgio Valla, 1501: Euclidean and Archimedean/Eutocian fragments;
– Bartolomeo Zamberti, 1505: problematic translations of Euclid;

[66] Guillaume Gosselin, ed. and trans., *L'arithmetique de Nicolas Tartaglia* (Paris: Gilles Beys, 1578).

[67] With one exception. In 1577 he had used *L* (for *latus*) for the first power, in the Tartaglia translation he uses ℞.

[68] Pierre Forcadel, *L'arithmeticque* (Paris: Guillaume Cavellat, 1556).

[69] See Mario Biagioli, "The Social Status of Italian Mathematicians, 1450–1600," *History of Science* 27 (1989): 41–95.

- Giovanni Battista Memmo, 1537: translations of Apollonios's *Conics* I–IV;
- Federico Commandino, 1566: another translation of Apollonios;
- Wilhelm Xylander, 1575: Diophantos;
- Federico Commandino, 1588: Pappos.

Various printings of Archimedean works (in Moerbeke's translation) were published by Luca Gaurico in 1503 and Tartaglia in 1543.

Finally, there were the editions of Greek texts:

- The Grynaeus edition of Euclid with Proclos's commentary in 1533;
- Ptolemy's *Almagest* in 1538;
- Archimedes in 1544.

In the later 16th century, a fairly full Greek mathematical canon was thus available.

The other aspect of the new cultural setting of mathematics in France and neighbouring areas was the reappearance of an agonistic mathematical culture centred on the solution of problems. It is reflected in a familiar story:[70]

> Viète's mathematical reputation was already considerable when the ambassador from the Netherlands remarked to Henry IV that France did not possess any geometricians capable of solving a problem propounded in 1593 by Adrian van Roomen to all mathematicians and that required the solution of a forty-fifth-degree equation. The king thereupon summoned Viète and informed him of the challenge. Viète saw that the equation was satisfied by the chord of a circle (of unit radius) that subtends an angle $2\pi/45$ at the center. In a few minutes he gave the king one solution of the problem written in pencil and, the next day, twenty-two more.

The story illustrates, firstly, that mathematical tournaments were not only a matter for mathematicians and mathematical dilettanti[71] but were by now a state affair; that would not have been the case a century before, or after. Secondly, that Viète (and his peers) were known to the court, and linked to it.

The present problem came from advanced algebra (though ultimately from trigonometry); in that respect it was not quite typical of the challenges exchanged between French dilettanti. Viète's *Book 8 of various responses about mathematical matters* from 1593 indi-

70 I quote Hubert L. L. Busard, "Viète, François," in *Dictionary of Scientific Biography*, vol. XIV (New York: Scribner, 1976), 22.

71 Viète, Fermat, and Descartes were all mathematical dilettanti; professional mathematicians of the time would be mathematical practitioners or teachers of these; at most professors like Oronce Finé. None of them were at the mathematical level of the best dilettanti, and these were the ones engaged in the challenges here concerned.

cates (since it contains "responses") what was typical:

- two intermediate proportionals;
- squaring and rectification of the circle and of circular segments, using Archimedean spirals and the quadratrix;
- construction of a regular heptagon;
- lunules; etc.
- In the end Viète deals with spherical trigonometry, a topic that had his special interest, and which may have helped him to recognize what was behind van Roomen's problem.

Why algebra?

So, the challenges that were *à la mode* were rooted in Greek geometry, which had become known thanks to the Humanist editions. Nothing seems to lead toward innovative interest in *algebra*.

But the challenges were *problems*, and algebra had always been known as a tool for solving problems.

Neither Viète nor Descartes say much about their inspiration except when they can castigate; in that respect they are similar to Borrel and Gosselin (and to many others). Descartes, however, left letters and other writings, and Beeckman wrote about him. So, we may start with him, and then see how Viète fits the pattern.

Peripherally we may observe at first that what Descartes makes in the *Geometrie* differs so completely from what Viète had done that there is no reason to expect even the slightest inspiration from him.

The *Geometrie* was purportedly meant as an illustrative appendix to Descartes' *Discours de la methode* from 1637. Actually, as shown by Christian van Randenborgh,[72] Frans van Schooten knew the finished manuscript already in 1632. But the *Discours* itself provides a first key. In part 1 (p. 18) he explains that when younger (namely at the Jesuit school) he had studied *un peu* logic and, "among the mathematics, the analysis of geometers and algebra," sciences that seemed to promise something for his project. But logic mostly served to express what one already knows (or does not know); analysis only deals with geometric figures; and the rules and signs of algebra (which Descartes had been taught on the basis of Clavius's book) have made it "a confused and obscure art that puts the mind in difficulty instead of a science that cultivates it."

This expresses the opinion Descartes had reached when writing the *Discours*. In any

[72] Christian van Randenborgh, "Frans van Schootens Beitrag zu Descartes' *Discours de la méthode*," *Mathematische Semesterberichte* 59 (2012): 225.

case, in a letter to Beeckman from 26 March 1619[73] he speaks of his intention to create "a wholly new science by which it will be possible in general to solve all questions that can be proposed dealing with any kind of quantities, continuous as well as discrete." In this letter, Descartes makes use of algebra in Clavius's style, employing his symbolism[74] – by the way without betraying any kind of familiarity with several unknowns (he may be right that he had studied the algebra textbook just *un peu*).

After a long separation Descartes met Beeckman again on 8 October 1628. This time Beeckman's *Journal*[75] informs us that Descartes has invented a "general algebra;" but his notation is still that of Clavius.

That, as we know, has changed in the *Geometrie*. Between 1628 and 1632, Descartes has thus learned about the use of multiple unknowns. He might of course have invented them on his own, but there is evidence that Descartes learned from Beeckman – shortly afterwards, thus Beeckman, Descartes proposed either to send him from Paris his *Algebra* for correction or to visit him for collaboration (the former must have happened; the manuscript in question has disappeared, apart from likely extracts copied by Beeckman[76]).

Since Beeckman owned a copy of Mennher's *Arithmetic*, this book is a likely source for Descartes new insight (still unless we believe in an independent discovery); inspiration from Peletier, Borrel or Gosselin seems unlikely.

Descartes himself has this to say in the *Geometrie:*[77]

> When wishing to solve some problem, one should first look at it as already solved, and give a name to all the lines that seem to be needed in order to construct it, those that are unknown as well as the others. Then, without making any difference between these known and unknown lines, one should run through the difficulty according to the order it shows, the most natural of all, in which way they depend mutually on each other, until the point where one has found a way to express one and the same quantity in two ways: which is called an equation.

Descartes (as already Stifel) thus does not look for a minimal set of unknowns – what appears to play a role gets a name. Once he applies algebra with several unknowns to geometric problems, quasi-abstract names (that is, not specified numbers but names linked

73 Charles Adam and Paul Tannery, eds., *Oeuvres de Descartes*. Vol. X: *Physico-mathematica. Compendium musicae. Regulae ad directionem ingenii. Recherche de la vérité. Supplément à la correspondance* (Paris: Léopold Cerf, 1908), 154–160.

74 Cf. p. 155, note d in the edition.

75 Cornelis De Waard, ed., *Journal tenu par Isaac Beeckman de 1604 à 1634*, vol. III (La Haye: Nijhoff, 1939–1953), 94–97.

76 Ibid., vol. IV, 135–139. The fragment does not elucidate the questions that concern us here.

77 René Descartes, *Discours de la methode* (Leiden: Ian Maire, 1637), 300.

to specific entities appearing in a diagram) would be used for everything pertinent – no distinction being made between what would turn up as coefficients and constants in the equations and what would turn up as unknowns.

Slightly later, without explanation Descartes starts following the principle to use letters from the end of the alphabet (first *z*, then if needed *y*, etc.) for the unknown magnitudes and letters from its beginning for those that are known.

The words "all the lines that seem to be needed" are significant. Traditional lettered diagrams mostly give letter-names to *points*, not to the segments. Algebra, however, applies to quantities, and these are the segments. Descartes therefore tacitly changes the way lettering is made; already Stifel had done so, and Viète does so too.

Viète, like Descartes, speaks badly about the algebra he has inherited, namely as "a new art, or rather so old and so defiled and polluted by barbarians that I have found it necessary to bring it into, and invent, a completely new form," in the words of his *Zetetics*.[78] He may well have adhered to the widespread Humanist claim that algebra was invented by Diophantos and then distorted by the Arabs, but he says nothing more precise. He takes over Diophantos's (that is, Xylander's) names for the powers, and borrows the idea of *analysis* from Greek authors. He also mentions Cardano's *Practica arithmeticae* twice, censuring (fols. 9ᵛ, 10ʳ). That is all.

He is unlikely not to have known Cardano's *Ars magna*. Equally implausible is that he should not have known Gosselin. Viète was engaged in spherical geometry, and he is thus likely to have known Mennher's book about that topic; possibly therefore also his *Arithmetique seconde*.

Irrespective of which predecessors have inspired Viète's use of several unknowns, his own first step as he describes in the *Isagoge* is similar to that of Descartes:[79]

> Magnitudes, those which are known as well as those which are asked for, should be combined and compared, adding, subtracting, multiplying and dividing, always observing the law of homogeneity.

For both, the decision to apply algebra with several unknowns to geometric problems entailed as an obvious consequence the abstract coefficients. *It came automatically as a gift not asked for.* Both were self-conscious to the point of being haughty, none of them afraid to claim to have done completely away with the filth, pollution and confusion of the algebra they had inherited. But precisely *this* point, which in later historiography is seen as the most important aspect of the New Algebra, they hide within subordinate clauses, "those which are known as well as those which are asked for" respectively "those that are

78 François Viète, *Zeteticorum libri quinque* (Tours: Jamet Mettayer, 1591), fol. 2ᵛ.
79 François Viète, *In artem analyticem isagoge* (Tours: Jamet Mettayer, 1591), fol. 7ʳ.

unknown as well as the others."[80]

Most of the other characteristics that separate the New Algebra from that of al-Khwārizmī had accumulated gradually since the 14th century, as we have seen. One was not yet ripe: the general parenthesis. Descartes as well as Viète still made use of formal fractions. Descartes introduced the new root sign which made nesting much easier; he also used braces to define a new special-purpose parenthesis (containing a polynomium that serves as a factor), but used it only for that purpose. Oughtred, Newton and others started using a vinculum (a stroke above a composite expression) for the same purpose, but it remained a special-purpose tool. Only in infinitesimal operations would Wallis and Newton go cautiously beyond this boundary, soon to be followed more courageously by Johann Bernoulli and Leibniz; the abundance of new functions and the explicit function concept created a need to allow composite expressions as arguments. Then, after the mid-18th century, ordinary algebra learned from its infinitesimal offspring to see the parenthesis as so obvious that it needed no explanation. This is another complicated story, which I have told elsewhere.[81]

80 Viète, *In artem analyticem isagoge* fol. 5^r, does give the new type a name, *speciosa*, "which shows things by the species or forms of things, possibly by the elements of the alphabet." The name reappears on fol. 8^r and in the titles of *Ad Logisticem speciosam notae priores* and ... *posteriores*, announced on fol. 1^v as part of the total algebraic project. On fol. 8^r it is said to be simple whereas Diophantos' *logistica numerosa* is so difficult that it has come to be much admired. The term does not serve as a fanfare of victory but as a sober description.

81 Høyrup, "How Did the All-Purpose Parenthesis Come About in European Algebra?"

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