Johann Scheubel, Wilhelm Xylander and the numerical treatment of Euclidean geometry in early modern Europe

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Abstract

The Latin and German commentaries on the first six books of the *Elements* by Johann Scheubel (1494–1570), professor at the University of Tübingen, and Wilhelm Xylander (1532–1576), professor at the University of Heidelberg, stand out within the sixteenth-century Euclidean tradition by their extensive use of arithmetic and cossic algebra (in the tradition of Christoff Rudolff and Michael Stifel) in their respective exposition of Euclid's geometrical propositions, which was primarily connected with their Protestant background and pedagogical context. By analysing Scheubel and Xylander's commentaries and their numerical approach to Euclid's geometrical propositions, this article aims to offer an insight into the evolution of the arithmetization of Euclidean geometry in early modern Europe.

Keywords

Euclid's Elements, arithmetization of geometry, Johann Scheubel, Wilhelm Xylander, Protestant universities

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Introduction

Euclid's *Elements*, an ancient Greek treatise of geometry and arithmetic, was often regarded in the early modern period as the standard source for the teaching of mathematics (and geometry in particular) at an elementary level, for which reason it was increasingly edited, translated and commented from its first printed edition in 1482 (i.e. Erhardt Ratdolt's edition of Campanus of Novara's thirteenth-century commentary).[1]

Since the first Latin translations of the *Elements* in the twelfth century, it was acknowledged that, in this text, Euclid's *modus operandi* was properly demonstrative, deriving the truth of propositions from universally admitted first principles (axioms), and dealing with numbers and magnitudes in a purely abstract manner, without any reference to concrete objects, instrumental procedures or applications.[2] In the *Elements*, magnitudes were also kept separate from numbers, as the quantitative properties or relations of magnitudes were not expressed in numerical terms, by attributing specific measures to lines and figures according to a given unit of length, area and volume.[3] These remained abstract and quantitatively unspecified, their equality or proportion being determined through rational deduction only. A similar approach was adopted for numbers in the arithmetical books of the *Elements*, since these were then represented by non-quantified lines of different lengths rather than by specific numerical examples.[4]

Moreover, although the principles of Eudoxian theory of ratios and proportions taught in the *Elements* were applicable to both numbers and magnitudes, Euclid had chosen to propose a separate treatment of this doctrine for magnitudes (in Book V) and numbers (in Books VII-IX).[5] Although the reason for this separation was not made explicit by Euclid,[6] it is most likely linked with the opposition between discrete and continuous quantity, that is, between a type of quantity composed of indivisible units,

1 Campanus de Novara, *Preclarissimum opus elementorum Euclidis megarensis una cum commentis Campani perspicacissimi in artem geometriam* (Venezia: Erhard Ratdolt, 1482).

2 This acknowledgement appears for instance in Dominic Gundisalvi's *De divisione philosophiae* in which the nature of theoretical geometry is mainly illustrated through examples drawn from the *Elements*. Gundisalvi, *De divisione philosophiae*, in Ludwig Baur, *Dominicus Gundisalvi. De divisione philosophiae* (Münster: Aschendorff, 1903), 106–107.

3 This was generally displayed in the sixteenth century in editions presented as faithful to its Greek tradition, such as those of Bartolomeo Zamberti, *Euclidis megarensis [...] elementorum libros XIII cum expositione Theonis insignis mathematici* (Venezia: Johannes Tacuinus, 1505) or Conrad Dasypodius, *Euclidis quindecim elementorum geometriae primum: ex Theonis commentariis Graecè, & Latine* (Strasbourg: Christianus Mylius, 1564).

4 Bernard Vitrac, *Euclide. Les Éléments. Livres V–VI: Proportions et similitude. Livres VII–IX: Arithmétique* (Paris: Presses universitaires de France, 1994), 280–282.

5 Vitrac, *Les Éléments. Livres V–VI*, 15–19.

6 Thomas L. Heath, *The Thirteen books of Euclid's Elements* (New York: Dover, 1956), vol. 1, 112–113.

as numbers were thought to be in ancient Greek arithmetic (and thus also in the *Elements*[7]), and a type of quantity that is continuous and therefore infinitely divisible, as geometrical lines and figures (plane and solid). According to this opposition, which gave rise to the distinction between arithmetic and geometry in ancient classifications of mathematics,[8] if numbers are always commensurable (due to the fact that they may all be resolved into the same indivisible unit) magnitudes may be either commensurable or incommensurable, as is the diagonal of the square with respect to its side, which may not be expressed as a whole number or as a ratio of numbers, but only as a radical number (i.e. $\sqrt{2}$).

In spite of the essentially abstract, rational and non-numerical treatment of magnitudes in Euclid's *Elements*, a number of editions of this treatise published from the mid-sixteenth century presented geometrical propositions in a hands-on and empirical rather than purely rational and demonstrative manner, prioritising utility, efficiency and user-friendliness over logical rigour and scientific universality.[9] These editions aimed at offering a version of Euclid's treatise that would be more accessible to young students and readers (including artisans and merchants) who were less familiar with the style and content of ancient mathematics, but who might see in the *Elements* a source of information on procedures useful to their art as well as a knowledge that could give them access to a higher status.[10] In this context, the numerical treatment of magnitudes played a key role, as it enabled a first-hand verification of theorems through specific examples, thus reducing the importance of logic-based demonstrations, and showed how certain Euclidean propositions may be applied in handling concrete quantities. This numerical approach to geometry, which was at the time characteristic of the treatises on practical geometry that focused on measuring procedures (instrumental and computational),[11]

7 Heath, *The Thirteen Books*, vol. 2, 277, Df. VII.2: "A number is a multitude composed of units."

8 Vitrac, *Les Éléments. Livres V–VI*, 19–32.

9 Antoni Malet, "Euclid's swan song: Euclid's *Elements* in early modern Europe," in *Greek Science in the Long Run: Essays on the Greek Scientific Tradition (4th c. BCE-17th c. CE)*, ed. by Paula Olmos (Cambridge: Cambridge Scholars Publishing, 2012), 205–234; Marta Menghini, "From practical geometry to the laboratory method: The search for an alternative to Euclid in the history of teaching geometry," in *Selected Regular Lectures from the 12th International Congress on Mathematical Education*, ed. by Sung Je Cho (Cham: Springer, 2015); Angela Axworthy, "The hybridization of practical and theoretical geometry in the sixteenth-century Euclidean tradition," *Journal of Interdisciplinary History of Ideas* 11/22 (2022): 1–104.

10 Thomas Morel, "Bringing Euclid into the mines: Classical sources and vernacular knowledge in the development of subterranean geometry," in *Translating Early Modern Science*, ed. by Sietske Fransen et al. (Leiden: Brill, 2017), 154–181.

11 On the nature of pre- and early modern practical geometry, see for instance Hervé L'Huillier, "Practical Geometry in the Middle Ages and the Renaissance," in *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, ed. by Ivor Grattan-Guinness (London:

diverged from the abstract and non-numerical Euclidean treatment of magnitudes.[12]

In this framework, the numerical expression of magnitudes also went beyond the use of whole numbers by including fractions[13] and irrationals. These were then generally called 'surd numbers' due to the fact that they cannot be numerically expressed with precision or in a direct manner, some being at least expressible as square or cubic roots, as the ratio of the diagonal to the side of the square, and others only as approximations, as the ratio of the diameter to the circumference of the circle. As would be expressed by Henry Billingsley, who published the first English commentary on Euclid's *Elements* in 1570 and adopted within it many numerical examples: "There are certaine lines magnitudes or quantities which can not be named and expressed by number, and therefore commonly are called Surd lines or magnitudes. […] the line BC shall be √32, which is a surd number, and can not be expressed by any determinate and certaine number, but only by this maner of circumlocution Roote square of 32. […] the proportion betwene them is irrational, confused, unknowen, uncertaine, and surd."[14]

The numerical expression of Book X, which deals with incommensurable magnitudes, thus did not only overthrow Euclid's separation between numbers and magnitudes in the *Elements*, but also raised the issue of the ontological status of irrationals (whether or not these may be considered as numbers in their own right) and of their admissibility in the context of a commentary on a Greek mathematical treatise as Euclid's *Elements*, which only admitted whole numbers and which only confined numbers to the arithmetical books.[15] In extreme cases, as that of the numerical treatment of Book X in the practice-oriented *Arithmetique* of the engineer and mathematician Simon Stevin (1585), this approach accompanied a full reassessment of the distinction between numbers and magnitudes founded on the rejection of the notion of number as discrete quantity and on the explicit admission of irrationals (at least as radical numbers) as proper numbers.[16]

Routledge, 1994), 185–191 and Eberhard Knobloch, "Géométrie pratique, géométrie savante," *Albertiana* 8 (2005): 27–56.

12 Henk Bos, *Redefining Geometrical Exactness Descartes' Transformation of the Early Modern Concept of Construction* (New York: Springer, 2001), 126, 128 and 135–136; Antoni Malet, "Renaissance notions of number and magnitude," *Historia mathematica* 33 (2006): 63–81.

13 Katherine Neal, *From Discrete to Continuous. The Broadening of Number Concepts in Early Modern England* (Dordrecht: Springer Science & Business Media, 2002), 16. Neal considers (ibid., 18) that, under certain conditions, Euclid may be considered as having admitted fractions. However, no concept of rational number appears in this framework, including to characterise ratios, unlike how it would be in the middle ages and early modern era through the concept of denomination, as will be presented later.

14 Henry Billingsley, *The Elements of Geometrie of the most auncient Philosopher Euclide of Megara* (London: John Daye, 1570), 127r–v. See Malet, "Renaissance notions."

15 Neal, *From Discrete to Continuous*, 14–22; Bos, *Redefining Geometrical Exactness*, 130–131.

16 Simon Stevin, *L'Arithmetique […] Avec l'Explication du Dixiesme livre d'Euclide* (Leiden: Chris-

This approach thus departed from the properly Euclidean approach to magnitude, which was associated with the rigour, universality and geometrical certainty that was often attributed to the *Elements* as a model of scientific knowledge by many ancient and modern philosophers and mathematicians.[17] It was, on the other hand, fully in line with the practice transmitted in treatises of practical geometry, practical arithmetic and algebra, notably in the handbooks of Italian *Maestri d'abbaco* and German cossic *Rechenmeister*.[18] The fact of attributing specific dimensions to magnitudes in the diagrams that accompany Euclid's proofs and using these dimensions to perform computational procedures, or even replacing magnitudes by numbers when dealing with ratios in Book V, thus connected the *Elements* with the computational practice of surveyors, merchants and bankers. This would also be the case in Stevin's *Arithmetique*, where content relating to commercial and financial arithmetic is proposed and where the author explicitly addressed a public of practitioners.[19]

In this regard, putting aside the special (and later) case of Stevin's treatment of Book X, the editions of Euclid's first six books by Johann Scheubel[20] and Wilhelm Xylander,[21]

tophe Plantin, 1585), 4v–5r and 32–33. On Stevin's approach to numbers, see Bos, *Redefining Geometrical Exactness*, 138–141; Neal, *From Discrete to Continuous*, 34–36; Malet, "Renaissance notions;" Jürgen Naets, "How to Define a Number? A General Epistemological Account of Simon Stevin's Art of Defining," *Topoi* 29 (2010): 77–86.

17 Angela Axworthy, *Le Mathématicien renaissant et son savoir. Le Statut des mathématiques selon Oronce Fine* (Paris: Vrin, 2005), 83–123.

18 The influence of practical mathematics on the transformation of the classical approach to numbers and magnitudes in the early modern era is shown in Neal, *From Discrete to Continuous.* On these Italian and German handbooks of practical arithmetic and algebra, see Albrecht Heeffer, "The Genesis of the Algebra Textbook: From Pacioli to Euler," *Almagest* 3/1 (2012): 26–61; Jens Høyrup, *The World of the Abbaco. Abbacus Mathematics Analyzed and Situated Historically between Fibonacci and Stifel* (Cham: Birkhäuser, 2024).

19 See, for instance, the title page and the address of the preface of *La Disme* (also published in Dutch in 1585 as *De Thiende*), in Stevin, *L'Arithmetique*, 132–133. On the practical orientation of Stevin's *La Disme*, and of his mathematical work more generally, see Neal, *From Discrete to Continuous*, 2.

20 Johann Scheubel, *Euclidis Megarensis, Philosophi & Mathematici excellentissimi, Sex libri priores, de Geometricis principiis, Graeci & Latini, unà cum demonstrationibus propositionum, absque literarum notis, veris ac proprijs, & alijs quibusdam, usum earum concernentibus, non citra maximum huius artis studiosorum emolumentum adiectis. Algebrae porro regulae, propter numerorum exempla, passim propositionibus adiecta, his libris praemissae sunt, eaedemque demonstratae* (Basel: Hervagius, 1550).

21 Wilhelm Xylander, *Die sechs erste Bůcher Euclidis, Vom anfang oder grund der Geometri, in welchen der rechte grund, nitt allain der Geometri (versteh alles kunstlichen, gwisen, und vortailigen gebrauchs des Zirckels, Linials oder Richtscheittes und andrer werckzeüge, so zu allerlai abmessen dienstlich) sonder auch der fürnemsten stuck und vortail der Rechenkunst, furgeschriben und dargethon ist, auß Griechischer sprach in die Teütsch gebracht, aigentlich erklärt, auch mit verstentlichen Exempeln, gründlichen Figurn geziert, dermassen vormals in Teütscher sprach nie gesehen worden* (Basel: Oporinus, 1562).

published in Basel in 1550 and 1562 respectively, stand out among the sixteenth-century editions[22] of the *Elements* that adopted a numerical treatment of geometrical principles and propositions by their extensive use of practical arithmetic and cossic algebra (in the tradition of Christoph Rudolff and Michael Stifel). As I will attempt to show, this was connected with their social, cultural and institutional background.

Although quite exceptional at the time, this approach became more frequent in the seventeenth-century Euclidean tradition across Europe, as the pedagogical agendas of the growing number of professors of mathematics evolved and as new mathematical developments took place.[23]

After presenting the lives and mathematical work of Scheubel and Xylander, this paper will show how these two authors, in their respective commentaries on the *Elements*, adapted and transmitted Euclid's geometrical propositions in a numerical manner through the integration of arithmetical and algebraic content. After that, a few words will be said on the practical character of this approach, before concluding on the role played by the context in which Scheubel and Xylander evolved, namely that of German Protestant universities, in fostering their numerical approach to Euclid.

The lives and works of Scheubel and Xylander

Johann Scheubel (or Scheybl) was born in 1494 in Kirchheim unter Teck and died in 1570 in Tübingen.[24] After early studies in the Latin school of Kirchheim, he enrolled in 1513 at the Faculty of Arts of Vienna. From 1532, he studied at the University of Leipzig and, in 1535, he enrolled at the University of Tübingen. After obtaining the title of *Magister* in 1540, he started giving lectures in mathematics (arithmetic and Euclid's geometry) in Tübingen in 1544 and became 'professor ordinarius' by 1550.[25] Evidence shows that he

22 For the sake of convenience, the term 'edition' includes here all the works designed to transmit the content of Euclid's *Elements*, thus including translations and commentaries.

23 Malet, "Renaissance notions"; Id., "Euclid's swan song."

24 On Scheubel's life and works, see Hermann Staigmüller, "Johannes Scheubel, ein deutscher Algebraiker des XVI. Jahrhunderts," *Abhandlungen zur Geschichte der Mathematik* 9 (1899): 429–469; Mary S. Day, *Scheubel as an Algebraist, Being a Study of Algebra in the Middle of the Sixteenth Century, Together with a Translation of and a Commentary upon an Unpublished Manuscript of Scheubel's Now in the Library of Columbia University* (New York: AMS Press, 1926); Ulrich Reich, *Schriftenreihe des Stadtarchivs Kirchheim unter Teck. 500 Jahre Johann Scheubel* (Kirchheim unter Teck: Gottlieb & Osswald, 1994); Id., "Johann Scheubel (1494–1570): Geometer, Algebraiker und Kartograph," in *Der „mathematicus": zur Entwicklung und Bedeutung einer neuen Berufsgruppe in der Zeit Gerhard Mercators*, ed. by Irmgard Hantsche (Bochum: Brockmeyer, 1996), 151–182; Id., "Scheubel, Johann," in *Neue Deutsche Biographie* 22 (2005): 709–710.

25 Rudolph von Roth, *Urkunden zur Geschichte der Universität Tübingen aus den Jahren 1476 bis 1550* (Tübingen: H. Laupp, 1877), 236–237 and 658. The title of *professor ordinarius* appears

was still teaching mathematics in Tübingen in 1562[26] and that he bequeathed his instruments and library to the University at his death in 1570.[27]

Beside his 1550 Latin edition of the *Elements* (Books I-VI) and a German edition of the arithmetical books (Books VII-IX) published in Augsburg in 1555,[28] Scheubel wrote commentaries on the *De Numeris Datis* of Jordanus Nemorarius[29] and on Robert of Chester's Latin translation of al-Khwārizmī's *Algebra*,[30] as well as two treatises of practical arithmetic published in Leipzig in 1545 and in Basel in 1549.[31] He also published in Paris in 1551 and 1552 a treatise of algebra, *Algebrae compendiosa facilisque descriptio*,[32] which was first published in 1550 as an introduction to his Latin edition of Euclid.[33] The latter represented one of the earliest treatises of algebra published in France. He also published in 1553 an edition of Jacques Lefèvre d'Étaples's introduction to Boethius' *De institutione arithmetica*.[34] As shown by the inventory of his library and manuscripts, Scheubel also worked on a Latin version of his exposition of Euclid's arithmetical books, as well as on Book X (in Latin and German) and Books XI-XV of the *Elements* (in Latin).[35] He is also

notably in the title page of his Latin edition of Euclid: Scheubel, *Euclidis elementa*, title page: "*in inclyta Academia Tubingensi Euclidis professore ordinario*."

26 Staigmüller ("Johannes Scheubel," 437) notes that Scheubel requested an increase of salary from the Senate in 1562 (as in 1551).

27 Barnabas Hughes, "The private library of Johann Scheubel, sixteenth-century mathematician," *Viator* 3 (1972): 417–432.

28 Johann Scheubel, *Euclidis elementa* and Id., *Das sibend, acht und neünt Bůch, des hochberůmbten Mathematici Euclidis Megarensis, in welchen der operationen und regulen aller gemainer rechnung, ursach grund und fundament, angezaigt wirt, zů gefallen allen den, so die kunst der Rechnung liebhaben, durch Magistrum Johann Scheybl, der lôblichen universitet zů Tübingen des Euclidis und Arithmetic Ordinarien, auß dem latein ins teütsch gebracht, unnd mit gemainen exempeln also illustrirt unnd an tag geben, das sy ein yeder gemainer Rechner leichtlich verstehn, unnd jene nutz machen kan* (Augsburg: Ottmar, 1555).

29 On this work, which remained unpublished, see Barnabas Hughes, "Johann Scheubel's revision of Jordanus de Nemore's *De numeris datis*: An analysis of an unpublished manuscript," *Isis* 63/2 (1972): 221–234.

30 Louis Karpinski, *Robert of Chester's Latin translation of the Algebra of al-Khowarizmi* (New York: Macmillan, 1915).

31 Johann Scheubel, *De numeris et diversis rationibus seu regulis computationum opusculum* (Leipzig: Michael Blum, 1545) and Id., *Compendium arithmeticae artis* (Basel: Johannes Oporinus, 1549).

32 Scheubel, *Algebrae compendiosa facilisque descriptio* (Paris: Guillaume Cavellat, 1551).

33 Scheubel, *Euclidis elementa*, 1–76.

34 Johann Scheubel, *Iacobi Fabri Stapulensis in Arithmetica Boëthi epitome, unà cum difficiliorum locorum explicationibus & figuris (quibus antea carebat) nunc per Ioannem Scheubelium adornatis & adiectis. Accessit Christierni Morssiani Arithmetica practica* (Basel: Henri Estienne, 1553).

35 Hughes, "The Private Library," 425: "1. Commentarius in decem librorum Euclidis teutsch Scheubelii imperfectus et latine bis. 2. Idem Scheubelius in libros Arithmeticos 7, 8, 9 Euclidis latine bis." Scheubel's commentary on Books XI–XV, written in 1561, is situated in Ms. Biblio-

credited for having produced the oldest map of the Duchy of Nüremberg in 1559.[36]

Wilhelm Xylander (Holtzman or Holtzmann) was born in 1532 in Augsburg and died in Heidelberg in 1576.[37] He came from a modest family, but received the support of the humanist Sixtus Birk (or Birck) for his early education.[38] In 1549, he enrolled at the University of Tübingen, where he received the bachelor degree in 1550. Although his biographer Fritz Schöll claims that he was self-taught in mathematics, he may have attended Scheubel's mathematics lectures at Tübingen. In 1555, Xylander started working on his German translation of Euclid's *Elements*, which was published in 1562.[39] In 1557–1558, he studied at the university of Basel, where he received a Master's degree. He then went to the University of Heidelberg where the taught Greek (1558–1563), mathematics (1562–1563) and logic (1563–1576). In 1561, he was appointed librarian of Elector Friedrich III's library. He was also dean of the Arts Faculty of Heidelberg in 1563–1564 (and in 1571), as well as rector of the university in 1564–1565.

Beside his 1562 German edition of Euclid and a Latin translation of Diophantus' *Arithmetica* published in 1575,[40] Xylander translated and edited various Greek and Latin works by Aristotle, Strabo, Diodorus of Sicily, Plutarch and Pausanias, among others, as well as Michael Psellos' *De quattuor mathematicis scientiis*.[41] He also authored a mathematical compendium entitled *Opuscula Mathematica* published in 1577 in Heidelberg,[42] as well as a versified eulogy of the astronomical clock of the cathedral of Strasburg, dedicated to its

teca Apostolica Vaticana, Pal. lat. 1350, 1–320, under the title *Tertius tomus geometriae Euclidis, de quantitate continua in solidis*.

36 Ulrich Reich, *Johann Scheubel und die älteste Landkarte von Württemberg 1559* (Karlsruhe: Hochschule für Technik, 2000).

37 On Xylander's life and works, see Fritz Schöll, "Xylander, Wilhelm," *Allgemeine Deutsche Biographie* 44 (1898): 582–593; Dorothee Gall, "Xylander, Guilielmus," *Brill's New Pauly Supplements I Online*, vol. 6; Dagmar Drüll, *Heidelberger Gelehrtenlexikon 1386–1651* (Berlin; Heidelberg: Springer, 2002), 562–563.

38 On Sixt, see Alfred Hartmann, "Birk, Sixt (Xystus Betul[e]ius)," *Neue Deutsche Biographie* 2 (1955): 256.

39 As indicated in the preface, which is addressed to the city council of Augsburg, Xylander had produced seven years earlier a translation of the first four books of the *Elements*, which he presented to these Augsburg notables. Xylander, *Euclidis elementa*, a2r: "Ich hab […] vor siben Jaren, die Vier erste biecher Euclidis vom Grund der Geometrj, auß Griechischer sprach in die Teutsche gebracht."

40 Xylander, *Diophanti Alexandrini Rerum arithmeticarum…* (Basel: Eusebius Episcopius, 1575).

41 Xylander, *Pselli, doctissimi viri, perspicuus Liber de quatuor mathematicis scientiis* (Basel: Johannes Oporinus, 1556). This work was dedicated to Ulrich Fugger, a humanist and member of the Fugger family of Augsburg.

42 Xylander, *Opuscula Mathematica. Aphorismi Cosmographici; De minutiis; De Surdorum Numerorum natura & tractactione; De usu Globi & Planisphærii tractatus* (Heidelberg: Jacob Müller, 1577).

designer, Conrad Dasypodius, and published in 1575 in Tübingen.[43]

The numerical treatment of Euclid's geometrical books by Scheubel and Xylander

In the following pages, each case will be analysed separately, starting with that of Scheubel. For each author, I will start with a general presentation of their commentary on Euclid's first six books of the *Elements*, describing its main goal, style, addressed readers and pedagogical apparatus. I will focus in particular on the numerical treatment of magnitudes proposed in each work, its distribution within Euclid's treatise, its place in the proof and/or commentary of specific propositions, the type of numbers and of arithmetical and/or algebraic teaching provided in this framework, and how the adoption of this numerical approach to Euclid's geometry is justified by their authors. I will also indicate how this treatment related to the tradition of practical mathematics, as represented both by practical geometry treatises and by the works of German *Rechenmeister*. For Scheubel in particular, a brief consideration of his 1555 German commentary on Euclid's arithmetical books (VII-IX), and of other mathematical works, will be provided as a means to manifest the way Euclid's *Elements* and the tradition of common *Rechenbücher* were interrelated in Scheubel's mathematical teaching.

Scheubel's numerical treatment of Euclid's Elements

Scheubel's 1550 Latin edition of Euclid's first six books of the *Elements*,[44] which included the Greek text of the principles and enunciations, as well as a pedagogical commentary, most likely reflected the content of his teaching at the University of Tübingen at the time he wrote this work.[45] In his rendering of Euclid's proofs, Scheubel chose to totally abandon the use of letters to label geometrical diagrams, which is fundamental to the style of Euclidean proofs, as it allows to connect the text with the diagram at each step of the constructive and deductive process.[46] As Scheubel claimed in the preface, this approach would make Euclid's propositions more accessible to early students, who, in his experience, would get confused by the abundance of letters in geometrical proofs.[47]

Scheubel's desire to make Euclid's *Elements* more accessible to beginners is also marked

43 Nicodemus Frischlin and Wilhelm Xylander, *Carmen de Astronomico Horologio Argentoratensi* (Strasbourg: Nicolas Wyriot, 1575).

44 On Scheubel's Euclid, see also Staigmüller, "Johannes Scheubel," 440–446.

45 This is confirmed by many passages of the preface, as Scheubel, *Euclidis elementa*, a3v and a4r.

46 The abandonment of diagrammatic letters was already announced in the title of Scheubel's edition, manifesting the importance this aspect played for him in this framework. Scheubel, *Euclidis elementa*, title page: "*Euclidis [...] Sex libri priores, de Geometricis [...] unà cum demonstrationibus propositionum, absque literarum notis.*"

47 Scheubel, *Euclidis elementa*, a3r–v.

by the inclusion of several pedagogical strategies that were typically found at the time in practical geometry treatises,[48] such as compass arcs in diagrams to facilitate the comprehension of the steps of the construction (Fig. 1),[49] dotted lines to help visualise impossibilities,[50] explicit references to the use of instruments in the text,[51] deictic sentences to invite the reader to draw information from the diagram,[52] but also the addition of briefer or more practical means of construction or demonstration to supplement Euclid's proof.

To consider compass arcs in Scheubel's diagrams, let us take the example of the diagram accompanying Prop. I.10 ("To bisect a given finite straight line"[53]) depicted in Fig. 1. While in prior editions of the *Elements* the diagram illustrating this proposition simply consisted in a bisected equilateral triangle,[54] namely the equilateral triangle that one is required to construct on the given straight line and then to divide equally (as per Prop. I.9) by bisecting the angle subtended by the given line, Scheubel here depicted the intermediary steps of the construction, that is, the intersection of the two circles required to bisect the angle of the triangle subtended by the given line. Yet, instead of drawing the whole circles, he only represented their intersecting arcs. This visual strategy, which represents the construction as performed in practice, as briefer to accomplish than through the tracing of circles, and which was often adopted in practical geometry treatises when teaching geometrical constructions, was also considered easier for students to understand, as the diagram is then less encumbered by a multitude of lines. This motivation for the use of such diagrams when teaching Euclid's *Elements* was brought forth much later by Christoph Clavius, who was one of the most influential sixteenth-century professor of mathe-

48 Preliminary research on such features within Scheubel's and Xylander's commentaries on Euclid, including on certain aspects of their numerical treatment of Euclid, may be found in Axworthy, "The Hybridization," 47–62 and 75–83.

49 On the function of compass arcs in geometrical diagrams, notably in early modern commentaries on Euclid's *Elements*, see Eunsoo Lee, "Let the Diagram Speak: Compass Arcs and Visual Auxiliaries in Printed Diagrams of Euclid's *Elements*," *Endeavour* 42/2–3 (2018): 78–98.

50 On the function of dotted lines, see Scheubel, *Euclidis elementa*, I.4, 86.

51 Ibid., I.1, 83 or I.9, 90: "Officio igitur circini."

52 Ibid., I.12, p. 93: "id quod ex sequenti cuiusque descriptione apparet."

53 Heath, *The Thirteen Books*, vol. 1, 267.

54 This is the case in Prop. I.10 of the following editions published before Scheubel's commentary on Euclid: Campanus, *Euclidis elementa*, a4v; Zamberti, *Euclidis elementa*, A4v; Luca Pacioli, *Euclidis Megarensis philosophi acutissimi mathematicorumque omnium sine controversia principis opera* (Venezia: Paganino Paganini, 1509), 7r; Oronce Fine, *In sex priores libros geometricorum elementorum Euclidis Megarensis demonstrationes* (Paris, Simon de Colines, 1536), 17; Niccolò Tartaglia, *Euclide megarense philosopho, solo introduttore delle scientie mathematice, diligentemente reassettato, et alla integrita ridotto* (Venezia: Venturino Ruffinelli, 1543), 18v; Joachim Camerarius, *Euclidis elementorum geometricorum libri sex conversi in latinum sermonem* (Leipzig: Valentinus Papa, 1549), 18.

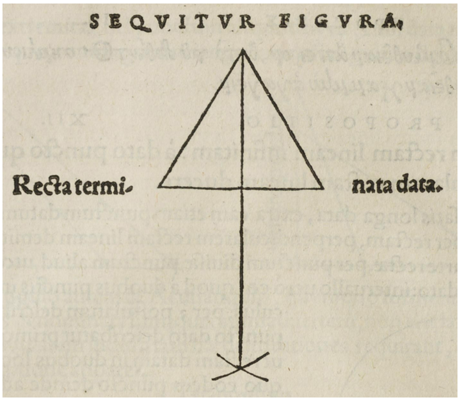


Fig. 1 – Scheubel, *Euclidis*, Prop. I.10, 91. Image source: Max Planck Institute for the History of Science, Berlin (Public Domain Mark 1.0).

matics and commentator of Euclid.[55] In this work, it is specifically related to a purely practical mode of teaching Euclidean constructions in the problems of Book I within commentary sections entitled *Praxis*.

While in Prop. I.10, Scheubel did not propose a construction different from that of Euclid, but simply displayed its steps in a more practical manner, one may find in Prop. I.3 ("Given two unequal straight lines, to cut off from the greater a straight line equal to the less"[56]) the addition of a different and properly mechanical construction to supplement Euclid's own construction. In this framework, Scheubel proposed three different constructions, starting from the easiest and most mechanical and ending with Euclid's construction, which is the most complex and theoretical.[57] In the first construction, unlike Euclid, Scheubel did not first request that a line equal to the shorter be placed at one extremity of the longer line (as per Prop. I.2), to then cut the longer line according to the length of the shorter by tracing a circle with a corresponding radius around their common extremity.[58] Rather he requested that the length of the shorter line be measured with a

[55] Christoph Clavius, *Euclidis elementorum libri XV: Accessit XVI*, in *Opera omnia*, vol. 1 (Mainz: Anton Hierat, 1611–1612), I.9, 35.

[56] Heath, *The Thirteen Books*, vol. 1, 246.

[57] Scheubel, *Euclidis elementa*, I.3, 84: "Est huius propositionis triplex operatio, seu fabrica."

[58] Heath, *The Thirteen Books*, vol. 2, p. 246: "*Given two unequal straight lines, to cut off from the greater a straight line equal to the less.* Let AB, C be the two given unequal straight lines, and let AB be the greater of them. Thus it is required to cut off from AB the greater a straight line equal to C the less. At the point A let AD be placed equal to the straight line C; and with centre A and distance AD let the circle DEF be described. Now, since the point A is the centre of the circle DEF, AE is equal to AD. But C is also equal to AD. Therefore each of the straight lines AE, C is

compass and then that the latter be moved onto the given point, having maintained the same opening and one of its feet having been placed on the point, to mark out the length of the shorter line on the longer line with its other foot.[59] Since it leaves out certain steps of the construction necessary to the demonstration (namely, the fact of first placing a line equal to the shorter at one extremity of the longer line according to the construction demonstrated in Prop. I.2) and because of its entire dependence on the compass, this mode of construction differs from that of Euclid by its mechanical and practical character. Its addition to Euclid's construction (along with a second construction of intermediary level of complexity[60]) allowed Scheubel to propose to students a construction easier to perform in practice, in addition to the less user-friendly construction of Euclid. In the constructions provided for Prop. I.3, the demonstrative part is also reduced to a minimum, consisting mainly in the indication of the principle and/or proposition on which it depends. Nevertheless, Scheubel was generally careful to maintain the classical Euclidean proof, even if he added many elements destined to help students understand them better or learn some of their further uses.

In addition to these pedagogical devices, numerical examples are employed when dealing with geometrical propositions, which was supported by the inclusion of a nearly eighty-page treatise of algebra, entitled *Brevis regularum algebrae descriptio una cum demonstrationibus geometricis*,[61] at the beginning of his edition of Euclid. In this treatise, Scheubel taught the fundamental notions and rules of algebraic computations, using at times concrete examples from trade, money-changing, financial partnerships, the military art and the measurement of physical magnitudes. The inclusion of this treatise of algebra in Scheubel's edition of the *Elements* confirms that the numerical treatment of Euclid's plane geometry and geometrical theory of proportions played a prominent role in his pedagogical agenda.[62] Scheubel applied indeed a numerical approach in all of Euclid's first six books except Book IV.[63] By contrast, most of the sixteenth-century editions that proposed

equal to AD; so that AE is also equal to C. Therefore, given the two straight lines AB, C, from AB the greater AE has been cut off equal to C the less."

59 Scheubel, *Euclidis elementa*, I.3, 84: "Prima, ut officio circini quantitas brevioris accipiatur: ea deinde in longiore, ab extremitate una incipiendo, puncto aliquo signetur: & factum erit negotium."

60 It was left aside here as irrelevant to the present study. For the complete treatment of this proposition by Scheubel, see Axworthy, "The Hybridization."

61 Scheubel, *Euclidis elementa*, 1–76. On this treatise, see Staigmüller, "Johannes Scheubel," 467; Day, *Scheubel as an Algebraist*.

62 The importance of the numerical approach to Euclid in Scheubel's transmission of the *Elements* is also marked by its dedication to members of the Fugger family, a powerful family of bankers of Augsburg.

63 Given that Book IV deals with inscriptions and circumscriptions of polygons within and about a circle, Scheubel may have regarded it as better suited to a fully constructive form of geometrical practice, as was used by painters and architects. On this type of practical geometry, see Lon

numerical examples to illustrate Euclid's geometrical teaching focused on Books II and V, and in rarer cases on Book X (when they went beyond Book VI).

Commentators of Euclid, already in the middle ages, indeed sometimes used numbers (mostly as whole numbers) to illustrate Book V (especially the definitions)[64] for pedagogical reasons on account of the fact that the Eudoxian theory of proportions taught in this book was considered applicable to all types of quantity (discrete and continuous), as was clearly acknowledged by Scheubel.[65] Since Book X also dealt with ratios, albeit between incommensurable magnitudes, it was also held acceptable in certain editions to use numerical examples with irrationals or surd numbers.[66] Book II, which mostly deals with quantitative relations between evenly or unevenly cut lines and the quadrangular areas on them, was sometimes considered as expressing algebraic identities as well as rules for arithmetical and algebraic operations, and was therefore sometimes treated in numerical terms (at least in Prop. II.1-10).[67] This was notably due to the influence of the arithmetical translation of these ten first propositions by Barlaam of Seminara in the fourteenth century.[68]

Yet, while most sixteenth-century editions that proposed a numerical treatment of Book V only used whole numbers and in some cases included fractions (in accordance with the late medieval notion of denominations of ratios[69]), mostly reserving irrationals

R. Shelby, "The Geometrical Knowledge of Mediaeval Master Masons," *Speculum* 47/3 (1972): 395–421.

64 Ratios of magnitudes expressed as whole numbers in Book V appear, mostly in the diagrams illustrating the proofs, in about a third of the published sixteenth-century editions of Euclid.

65 Scheubel, *Euclidis elementa*, Df. V.1, 226.

66 This was the case in the editions of Pacioli (1509), Tartaglia (1543), Pierre Montdoré (1551), Jean Magnien and Étienne Gracilis (1558), François de Foix-Candale (1566), Billingsley (1570) and Federico Commandino (1572).

67 This was the case in the editions of Jacques Peletier (1557), Dasypodius (1564), Pierre Forcadel (1564), Billingsley (1670), Commandino (1572) and Clavius (1574), in addition to Scheubel and Xylander.

68 Barlaam's treatment of Book II was taken up by Dasypodius, *Euclidis quindecim elementorum Geometriae secundum […]. Item, Barlaam monachi Arithmetica demonstratio eorum, quae in secundo libro elementorum sunt in lineis & figuris planis demonstrata* (Strasbourg: Christianus Mylius, 1564); Billingsley, *The Elements of Geometrie*, Book II; Federico Commandino, *Euclidis Elementorum libri XV. Unà cum scholijs antiquis* (Pesaro: Camillo Francischino, 1572), Book IX; Clavius, *Euclidis elementa*, Book IX. On Barlaam's arithmetical adaptation of Props. II.1–10, see Leo Corry, "Geometry and arithmetic in the medieval traditions of Euclid's *Elements*: A view from Book II," *Archive for History of Exact Sciences* 67 (2013): 637–705. On the arithmetical and algebraic interpretation of Euclid's Book II more generally, see Vitrac, *Les Éléments. Livres V-VI*, 366–376.

69 On this topic, see for example Sabine Rommevaux, "Aperçu sur la notion de dénomination d'un rapport numérique au Moyen Âge et à la Renaissance," *Methodos: Savoirs et textes* 1 (2001): 223–243.

for Book X (if they included Book X and dealt with it numerically), in Scheubel's commentary (as said) all of the commented books (except Book IV) were illustrated through numerical examples, using rational numbers or fractions (notably to express ratios in Book V) and irrational numbers (in Books I–II and VI), using various notations for square and cubic roots derived from the German cossic tradition.[70] In Book III, which merely includes numerical examples in the last two propositions (III.36–37), only whole numbers are used.[71]

Although Scheubel took care to explain that the definitions and propositions of Book V may be applied to numbers as well as to magnitudes[72] (wherefore he replaced the Euclidean term 'magnitude' by the more general term 'quantity' in this book), he did not only use numerical examples in Book V, as would Xylander for greater convenience,[73] but rather proposed, wherever it was possible, two types of examples: geometrical (using lines, as in the classical treatment of this book) and numerical (using integers and fractions).[74] He may have done so in response to the difficulties raised by earlier commentators of Euclid concerning the use of numbers in Book V, since the latter concerns ratios between both commensurable and incommensurable magnitudes[75] (some of which, as the ratio between the circumference and diameter of the circle, cannot even be expressed through radical numbers). But he may also have done so to simply manifest the universal scope of Book V, which extends to all types of quantities (discrete and continuous, commensurable and incommensurable).

To look in more details at Scheubel's numerical approach to Euclid's geometrical propositions, we may take the example of Prop. I.34 (Fig. 2–4), which states that "In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas"[76] (Fig. 2). It is the very first proposition of Euclid which Scheubel treated numerically,[77] but it is by no means exceptional in its style and content within his commentary on the *Elements*. As shown by the page exhibited in Fig. 2, while the figures accompanying the proof bear no letters (conforming to Scheubel's intention to free Euclid's

70 Scheubel notably took up Rudolff and Stifel's notations for square, cubic and fourth roots: √, ∛, ∜. On this notation, see Florian Cajori, *A History of Mathematical Notations*, vol. 1 (Chicago: The Open Court, 1928), 133–136 and 139–146.

71 Scheubel, *Euclidis elementa*, III.36–37, 203–205.

72 Ibid., Df. V.1, 226.

73 Xylander's approach to Book V is discussed later.

74 He also did so for the principles that were acknowledged as common to all branches of mathematics. Scheubel, *Euclidis elementa*, CN1–6, 81–82.

75 This problem had been, for instance, raised by Niccolò Tartaglia, in response to Campanus' use of numerical examples in his commentary of Book V. Tartaglia, *Euclide*, Df. V.11, 64v.

76 Heath, *The Thirteen Books*, vol. 1, 323–324.

77 It is certainly not coincidental, since this proposition was necessary for the demonstration of several propositions of Book II.

diagrams and proofs from the confusion of letters), they are attributed numerical values. This was very different from most other sixteenth-century editions that made use of numerical examples, as numerical diagrams were generally kept separate from the proofs and maintained within the commentary section (except perhaps in the case of Book V).

If the text of the proof itself remains devoid of numbers (except to refer to prior propositions), the commentary section, which is labelled *Appendix* in Prop. I.34, is entirely filled with computations and sometimes extends over several pages (three of which are presented in Figs. 2–4). In Prop. I.34, for that matter, the numerical commentary occupies more than four pages of the book.[78] Indeed, as we may see on Fig. 2 and 4, the *Appendix* starts just after the proof in the bottom half of p. 110 (Fig. 2) and ends at the bottom of p. 114 (Fig. 4), without interruption or addition of any other type of commentary. It is interesting to note in this regard that such sections were sometimes labelled *Numerorum praxis*,[79] which marks the operative and practical character of the numerical commentary.

Furthermore, Scheubel did not only attempt to illustrate the Euclidean proposition at hand through numerical examples, as was most often done in other editions that appealed to such pedagogical devices, but also extended the scope of his commentary by teaching computational rules, as well as methods to abbreviate the steps of the procedure, as they came in useful to solve geometrical questions by numerical means. In the framework of Prop. I.34, the second paragraph of the *Appendix* (Fig. 2) thus provides in general terms the method through which one can determine the areas of any triangle from the knowledge of their sides, which follows the method taught by Hero of Alexandria in his *Metrica*.[80]

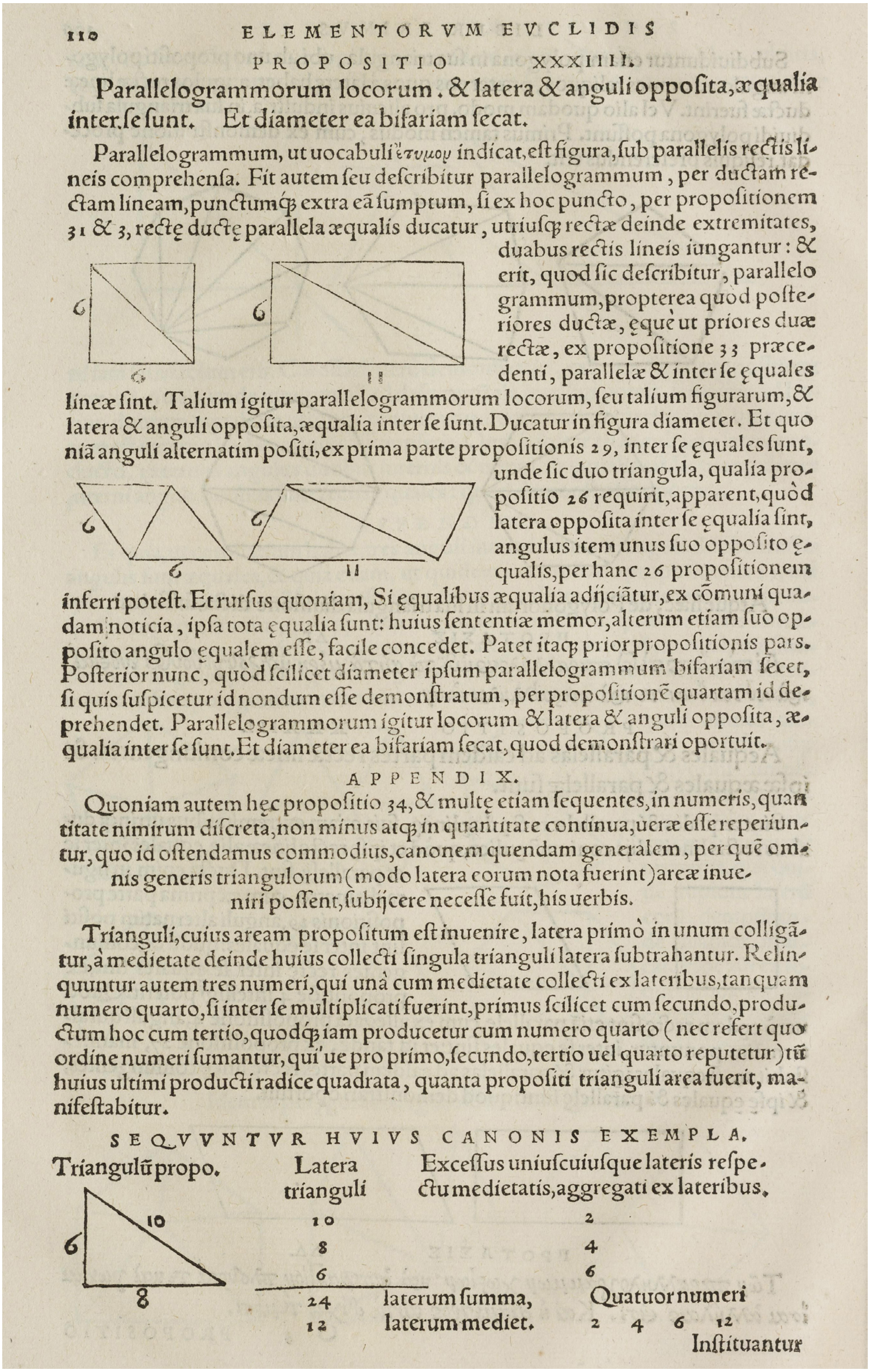
> Let first the sides of the triangles, whose area it is proposed to find, be added together; then let each of the sides of the triangle be subtracted from half of their total. Three numbers will remain, along with half of the added sides as a fourth number, which, if they are multiplied with each other, that is, the first is multiplied with the second, their product with the third, and the latter with the fourth (and the fact that the numbers are said to be the first, the second, the third or the fourth does not refer to the order according to which they should be added), then the square root of this last product will exhibit how much the area of the proposed triangle will be.[81]

78 Scheubel, *Euclidis elementa*, 110–114.

79 Ibid., I.36, 116: "Nunc quantum ad praxim numerorum;" I.40, 120: "Sequitur praxis numerorum" or I.41, 121: "Numerorum praxis."

80 Hermann Schöne, *Heronis Alexandrini Opera quae supersunt omnia*, Vol. III (Leipzig: Teubner, 1899), 18–25.

81 Scheubel, *Euclidis elementa*, I.34, 110.



PROPOSITIO XXXIIII.

Parallelogrammorum locorum. & latera & anguli oppoſita, æqualia inter ſe ſunt. Et diameter ea biſariam ſecat.

Parallelogrammum, ut uocabuli ἔτυμον indicat, eſt figura, ſub parallelis rectis lineis comprehenſa. Fit autem ſeu deſcribitur parallelogrammum, per ductam rectam lineam, punctumq; extra eã ſumptum, ſi ex hoc puncto, per propoſitionem 31 & 3, rectę ductę parallela æqualis ducatur, utriuſq; rectæ deinde extremitates, duabus rectis lineis iungantur: & erit, quod ſic deſcribitur, parallelogrammum, propterea quod poſteriores ductæ, ęque ut priores duæ rectæ, ex propoſitione 33 præcedenti, parallelæ & inter ſe ęquales lineæ ſint. Talium igitur parallelogrammorum locorum, ſeu talium figurarum, & latera & anguli oppoſita, æqualia inter ſe ſunt. Ducatur in figura diameter. Et quoniã anguli alternatim poſiti, ex prima parte propoſitionis 29, inter ſe ęquales ſunt, unde ſic duo triangula, qualia propoſitio 26 requirit, apparent, quòd latera oppoſita inter ſe ęqualia ſint, angulus item unus ſuo oppoſito ęqualis, per hanc 26 propoſitionem inferri poteſt. Et rurſus quoniam, Si ęqualibus æqualia adijciãtur, ex cõmuni quadam noticia, ipſa tota ęqualia ſunt: huius ſententiæ memor, alterum etiam ſuo oppoſito angulo ęqualem eſſe, facile concedet. Patet itaq; prior propoſitionis pars. Poſterior nunc, quòd ſcilicet diameter ipſum parallelogrammum biſariam ſecet, ſi quis ſuſpicetur id nondum eſſe demonſtratum, per propoſitionẽ quartam id deprehendet. Parallelogrammorum igitur locorum & latera & anguli oppoſita, æqualia inter ſe ſunt. Et diameter ea biſariam ſecat, quod demonſtrari oportuit.

APPENDIX.

Quoniam autem hęc propoſitio 34, & multę etiam ſequentes, in numeris, quantitate nimirum diſcreta, non minus atq; in quantitate continua, ueræ eſſe reperiuntur, quo id oſtendamus commodius, canonem quendam generalem, per quẽ omnis generis triangulorum (modo latera eorum nota fuerint) areæ inueniri poſſent, ſubijcere neceſſe fuit, his uerbis.

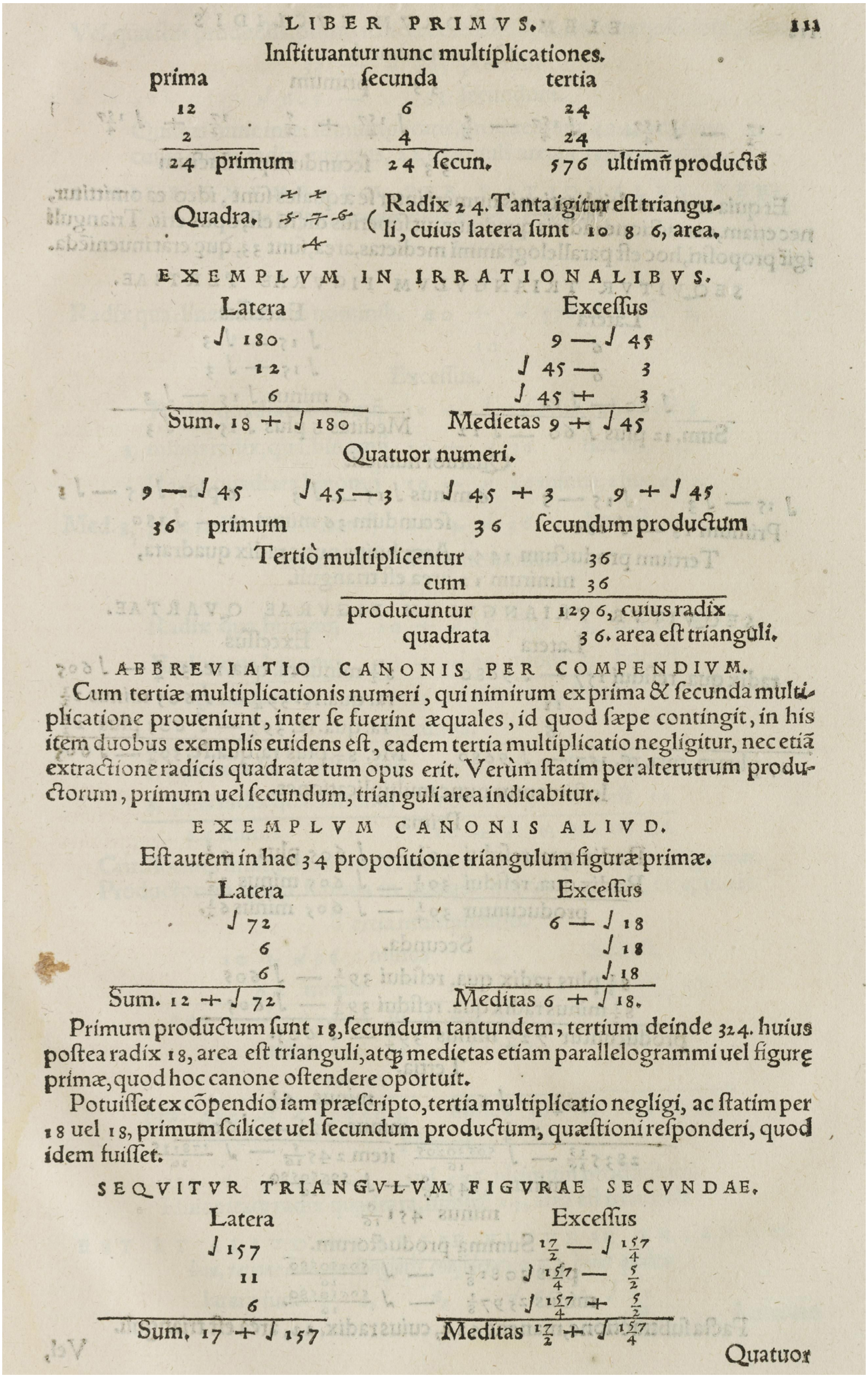
Trianguli, cuius aream propoſitum eſt inuenire, latera primò in unum colligãtur, à medietate deinde huius collecti ſingula trianguli latera ſubtrahantur. Relinquuntur autem tres numeri, qui unà cum medietate collecti ex lateribus, tanquam numero quarto, ſi inter ſe multiplicati fuerint, primus ſcilicet cum ſecundo, productum hoc cum tertio, quodq; iam producetur cum numero quarto (nec refert quo ordine numeri ſumantur, qui'ue pro primo, ſecundo, tertio uel quarto reputetur) tũ huius ultimi producti radice quadrata, quanta propoſiti trianguli area fuerit, manifeſtabitur.

SEQVVNTVR HVIVS CANONIS EXEMPLA.

Triangulũ propo.	Latera trianguli		Exceſſus uniuſcuiuſque lateris reſpectu medietatis, aggregati ex lateribus.
	10		2
	8		4
	6		6
	24	laterum ſumma,	Quatuor numeri
	12	laterum mediet.	2 4 6 12

Inſtituantur

Fig. 2 – Scheubel, *Euclidis elementa*, Prop. I.34, 110. Image source: Max Planck Institute for the History of Science, Berlin (Public Domain Mark 1.0).



Inſtituantur nunc multiplicationes.

prima		ſecunda		tertia	
12		6		24	
2		4		24	
24	primum	24	ſecun.	576	ultimũ productũ

Quadra. 576 (Radix 24. Tanta igitur eſt trianguli, cuius latera ſunt 10 8 6, area.

EXEMPLVM IN IRRATIONALIBVS.

Latera	Exceſſus
√ 180	9 — √ 45
12	√ 45 — 3
6	√ 45 + 3
Sum. 18 + √ 180	Medietas 9 + √ 45

Quatuor numeri.

9 — √ 45 √ 45 — 3 √ 45 + 3 9 + √ 45

36 primum 36 ſecundum productum

Tertiò multiplicentur 36

cum 36

producuntur 1296, cuius radix

quadrata 36. area eſt trianguli.

ABBREVIATIO CANONIS PER COMPENDIVM.

Cum tertiæ multiplicationis numeri, qui nimirum ex prima & ſecunda multiplicatione proueniunt, inter ſe fuerint æquales, id quod ſæpe contingit, in his item duobus exemplis euidens eſt, eadem tertia multiplicatio negligitur, nec etiã extractione radicis quadratæ tum opus erit. Verùm ſtatim per alterutrum productorum, primum uel ſecundum, trianguli area indicabitur.

EXEMPLVM CANONIS ALIVD.

Eſt autem in hac 34 propoſitione triangulum figuræ primæ.

Latera	Exceſſus
√ 72	6 — √ 18
6	√ 18
6	√ 18
Sum. 12 + √ 72	Meditas 6 + √ 18.

Primum productum ſunt 18, ſecundum tantundem, tertium deinde 324. huius poſtea radix 18, area eſt trianguli, atq; medietas etiam parallelogrammi uel figurę primæ, quod hoc canone oſtendere oportuit.

Potuiſſet ex cõpendio iam præſcripto, tertia multiplicatio negligi, ac ſtatim per 18 uel 18, primum ſcilicet uel ſecundum productum, quæſtioni reſponderi, quod idem fuiſſet.

SEQVITVR TRIANGVLVM FIGVRAE SECVNDAE.

Latera	Exceſſus
√ 157	$1\frac{7}{2}$ — √ $1\frac{57}{4}$
11	√ $1\frac{57}{4}$ — $\frac{5}{2}$
6	√ $1\frac{57}{4}$ + $\frac{5}{2}$
Sum. 17 + √ 157	Meditas $1\frac{7}{2}$ + √ $1\frac{57}{4}$

Quatuor

Fig. 3 – Scheubel, *Euclidis elementa*, Prop. I.34, 111. Image source: Max Planck Institute for the History of Science, Berlin (Public Domain Mark 1.0).

Laterum ſumma 12,plus $\sqrt{60} - \sqrt{12}$

Exceſſus igitur, atq; deinceps quatuor numeri,

$\sqrt{15} + \sqrt{3}$ $\sqrt{15} + \sqrt{3}$ 6 minus $\sqrt{15} + \sqrt{3}$ 6 plus $\sqrt{15} + \sqrt{3}$

Primum ſecundum productum

$18 + \sqrt{180}$ 36 minus $18 + \sqrt{180}$.

Tertium productum.

$648 + \sqrt{233280}$ minus $504 + \sqrt{233280}$

hoc eſt, 144. Area igitur trianguli 12, ut prius.

ALIVD ITEM TRIANGVLVM FIGVRAE quartæ, cuius quidem

Latera ſunt	Exceſſus igitur
Ra.qua.bi. $157 + \sqrt{9680}$	$8\frac{1}{2}$ minus ra.qua.bi. $39\frac{1}{4} + \sqrt{605}$
11	ra.qua.bi. $39\frac{1}{4} + \sqrt{605}$ minus $2\frac{1}{2}$
6	ra.qua.bi. $39\frac{1}{4} + \sqrt{605}$ plus $2\frac{1}{2}$
17 plus ra.qua.bi. $157 + \sqrt{9680}$	$8\frac{1}{2}$ plus ra. qua. bi. $39\frac{1}{4} + \sqrt{605}$

Producta,

primum	ſecundum
$72\frac{1}{4}$ minus $39\frac{1}{4} + \sqrt{605}$	$39\frac{1}{4} + \sqrt{605}$ minus $6\frac{1}{4}$

Inuentio producti tertij.

$72\frac{1}{4}$ minus $39\frac{1}{4} + \sqrt{605}$

$39\frac{1}{4} + \sqrt{605}$ minus $6\frac{1}{4}$

$2835\frac{13}{16} + \sqrt{\frac{50530205}{16}}$ item $245\frac{5}{16} + \sqrt{\frac{378125}{16}}$

minus $451\frac{9}{16}$

minus $2145\frac{9}{16} + \sqrt{\frac{59650580}{16}}$

Summa productorum.

plus $3081\frac{1}{8} + \sqrt{\frac{59650580}{16}}$

minus $2597\frac{1}{8} + \sqrt{\frac{59650580}{16}}$.

Hoc eſt, facta ſubtractione, 484.

Huius nunc tertij producti radix quadrata, nimirum 22, area eſt trianguli. Atq; hactenus dicta de triangulorum areis inueſtigandis ſufficiant. Sequitur

ΠΡΟΤΑΣΙΣ ΛΕ.

Τὰ παραλληλόγραμμα τὰ ἐπὶ τῆς αὐτῆς βάσεως ὄντα, καὶ ἐν ταῖς αὐταῖς παραλλήλοις· ἴσα ἀλλήλοις ἐστίν.

PROPOSITIO XXXV.

Parallelogramma ſuper eadem baſi conſtituta, atq; in eiſdem parallelis: æqualia inter ſe ſunt.

Poteſt huius propoſitionis figura geometrica tripliciter uariari. Aut enim parallelogrãmis ſuper una & eadem baſi, inter eaſdem item parallelas poſitis, alterum unius latus eſt diameter alterius, aut ea breuius, aut longius. Si primum, cum ex corollario propoſitionis præcedentis, Omne parallelogrammũ à ſua ipſius diametro bifariam

Fig. 4 – Scheubel, *Euclidis elementa*, Prop. I.34, 114. Image source: Max Planck Institute for the History of Science, Berlin (Public Domain Mark 1.0).

The Heronian formula to calculate areas of triangles may be expressed in modern terms as $A=\sqrt{s(s-a)(s-b)(s-c)}$, in which *s* corresponds to the semiperimeter of the triangle and *a*, *b*, *c* to each of the three sides of the triangle.

After providing the rule in general terms, Scheubel provides a first example with whole numbers (*Sequuntur huius canonis exempla*, bottom of Fig. 2 and top of Fig. 3), in which the considered triangle is depicted (with indication of the length of each side) next to the arithmetical operations required to compute its area. As we can see at the top of p. 111 (Fig. 3), Scheubel appeals in this first example to crossed-out numbers to mark out the steps of the calculation, as was generally done in treatises of practical arithmetic.

The example he proposes immediately after (*Exemplum in irrationalibus*, Fig. 3) involves more complex objects, notably irrational numbers (as the square roots $\sqrt{180}$ and $\sqrt{45}$) and binomials (as $9-\sqrt{45}$ or $\sqrt{45}+3$). Following this example, Scheubel provides a method to perform the computation in a briefer, and thus more practical, manner ('Abbreviation of the rule by economy,' or in Latin *Abbreviatio canonis per compendium*, as shown in the bottom half of p. 111, Fig. 3). This method consists in leaving aside the steps corresponding to the third multiplication and the extraction of the square root, when the product of two of the numbers to multiply (as here $\sqrt{18}$ and $\sqrt{18}$, corresponding to $(s-b)(s-c)$ in the modern expression of the formula) equals the product of the other two numbers (here $(6-\sqrt{18})(6+\sqrt{18})$ or $s(s-a)$).

In the following three pages of Scheubel's commentary, the same rule is illustrated through a multitude of examples (nine in total) with different types of numbers and numerical expressions, from the simplest to the most complex. Fig. 4, which exhibits the last page of the commentary on Prop. I.34 (p. 114), shows that in the last example proposed by Scheubel (*Aliud item triangulum figurae*), one may find fractions (as 72¼), radical numbers (as $\sqrt{605}$), binomials (as $39\frac{1}{4}+\sqrt{605}$), but also square roots of fractions (as $\sqrt{\frac{50530205}{16}}$) and square roots of binomials (as Ra. qua. bi. $157+\sqrt{9680}$). In this framework, arithmetical operators are presented in both symbolic and rhetorical forms for a better legibility, allowing to distinguish binomials from operations on binomials (as 72¼ minus $39\frac{1}{4}+\sqrt{605}$). As we may see, after the first numerical example provided to illustrate the rule (starting on p. 110 in Fig. 2), no geometrical representation of triangles is given for the other examples. At the end of this long series of examples, Scheubel simply concludes: "And what has been said so far on the determination of the areas of triangles will suffice."[82]

As Scheubel wrote in the introduction of his *Appendix* on this proposition (Fig. 2), the fact of dealing with propositions such as Prop. I.34 numerically is justified by the fact that it is applicable to both numbers and magnitudes: "Since this thirty-fourth proposition, and also many of those that follow, are found to be true for numbers, that is, for discrete quantity, as much as for continuous quantity, it was necessary, in order to show this more

[82] Ibid., I.34, 114.

appropriately, to provide a general rule by which the areas of any type of triangles may be found (provided their sides are known) in the following terms."[83] As he explains then (and as we saw above), the arithmetical rules that he taught in this framework were not simply applied to numbers, in addition or in parallel to the geometrical demonstration (e.g. to manifest the connections between arithmetic and geometry), but were directly applied to the specific dimensions of particular magnitudes, and more specifically to the areas of triangles (whether visually represented or not), overthrowing thereby the separation that had been set between numbers and magnitudes in the *Elements*.

The properly algebraic content Scheubel introduced in his commentary on Euclid, beyond the *Algebrae descriptio*, is to be found, as expected, in Book II, and notably in Prop. II.4, which states that "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments,"[84] and which may thus be taken to express (in modern terms) the following identity: $x^2 = a^2 + b^2 + 2ab$. The *Appendix* to Prop. II.4, which is three and a half pages long, is introduced thus: "Computists (*logistici*) employ this proposition rather frequently in the rules of algebra. For by means of it they usually demonstrate, among other things, the addition of irrational numbers, and then the equation in which, when three quantities are proposed in natural order or in equal means (*naturalis ordinis vel aequalibus medijs)*, the two of greater denomination is equal to the third, which has the smallest denomination."[85]

While the *Algebrae descriptio* is rarely cited in the commentary on Euclid, it is mentioned when explaining the use of Prop. II.4 to teach rules for the resolution of equations,[86] which confirms that Scheubel did not publish his treatise of algebra and his edition of the *Elements* together out of convenience, but initially intended for them to be read together. As Scheubel mentions in the commentary on Euclid, among the types of equations presented in his *Algebra*, it is the second type of the first form or canon (among the three distinguished canons), i.e. *Prima + radix = Numero* (or in modern terms $ax^2 + bx = c$),[87] that is founded on Euclid's Prop. II.4.[88] Euclid's proposition is itself cited in the *Algebra* when presenting this type of equation.[89]

As was previously mentioned, Scheubel also published a German adaptation of Books VII-IX of the *Elements*,[90] which differs from the edition of Books I-VI by its less scholarly

83 Ibid., I.34, 110.
84 Heath, *The Thirteen Books*, vol. 1, 379.
85 Scheubel, *Euclidis elementa*, II.4, 143.
86 Ibid., II.4, 144.
87 The other two correspond to $bx + c = ax^2$ and $ax^2 + c = bx$. Scheubel, *Algebrae descriptio*, in *Euclidis elementa*, 26–27 and II.4, 144.
88 Ibid., 144.
89 Scheubel, *Algebrae descriptio*, in *Euclidis elementa*, 28.
90 On this work, see also Staigmüller, "Johannes Scheubel," 447–450.

rendering (beside the properly arithmetical character of its subject[91]), as by its more explicit references to the knowledge of *Rechenmeister*,[92] and notably to the work of Christoff Rudolff.[93] This edition also taught, in addition to Euclid's theory of numbers and the Latin names of the employed German arithmetical terms,[94] various numerical examples and computational rules, with crossed-out numbers to mark out the steps of the computations, as in the 1550 Latin edition and as in common arithmetic handbooks.

Yet, while in Scheubel's Latin (and partly Greek) edition of Euclid, examples demonstrating concrete applications remained confined to the treatise of algebra, in the German edition, such examples, which mostly relate to the practice of merchants, bankers or book-keepers,[95] are directly inserted in the commentary of Euclid's propositions, sometimes extending over more than ten pages.[96] Numbers are then often expressed in concrete counting units, such as currencies (e.g. Gulden, Schilling, Pfennig) and measuring units (as ells), their regional differences being sometimes taken into consideration.[97]

This German edition confirms that Scheubel, in his general approach to Euclid's mathematical work (hence also in the 1550 Latin edition), intended to offer a different teaching of the *Elements* from that of his predecessors (as Oronce Fine, whose 1536 edition of Euclid was part of Scheubel's library[98]), which would be both more accessible and more useful to students and practitioners. In doing so, he also transmitted certain arithmetical and algebraic rules, and some of their concrete applications, through a different and more scholarly approach than through a mere *Rechenbuch*. These two editions of Euclid, whether in Latin or German, additionally allowed Scheubel to exhibit the theoretical foundation

[91] Even if Scheubel, in the commentary on Prop. VII.19 (Scheubel, *Das sibend, acht und neünt Bůch Euclidis*, 63–69), also quoted certain propositions from Book V and VI (namely, V.11, V.9, V.7 and VI.13).

[92] Scheubel, *Das sibend, acht und neünt Bůch Euclidis*, Introduction, 1: "In diesen dreyen bůchern findet man erstlich etlicher wörter, so *bey allen Rechenmaistern* in gemainen brauch seind" or 2: "Man findet auch weyter beyleüfftig die ursach, warumben *der Rechenmayster* im sůchen des quadrats wurtzel [...]" (my emphasis).

[93] Ibid., VIII.5, 143: "welches der fürnem und kunstreich Rechenmaister, Christoff Rudolff selig, in seinem gmainen Rechenbůchlen."

[94] Ibid., Df. VII.1–3, 3: "Ains, im latein *Unitas* [...] Die zal, im latein *Numerus* [...] Ain tail, im latein *Pars*."

[95] Ibid., VII.13, 39–40: "Das der Kauffman oder Handelßman [...]." See also 69: "Ainer kaufft 5 eln tůch zů einem rock, ist brait 1 1/2 eln."

[96] In Prop. VIII.5 (ibid., 131–146), such examples cover fifteen pages.

[97] Scheubel, *Das sibend, acht und neünt Bůch Euclidis*, 58–61: "Würtenberger müntz [...] Nůrnberger müntz [...] Meysnisch müntz [...] Baierisch und schwartz müntz [...] Osterreichisch müntz [...] Oetsch müntz, etc."

[98] Fine, *Euclidis elementa*; Hughes, "The Private Library," 422: "Orontii Geometrica in Euclidem." Although Fine's Euclid does include numerical examples, it overall remains faithful to the style of classical editions of Euclid, as that of Zamberti.

of a part of his arithmetical and algebraic teaching and thus to raise the status of common *Rechenkunst* in the hierarchisation of mathematical arts while expanding the readership of both traditions. It should be noted in this regard that his 1545 treatise of arithmetic, namely the *De numeris et diversis rationibus seu regulis computationum opusculum*, which was published in Latin in a more scholarly style than his German edition of Euclid's arithmetical books, included examples of computational practices applied to concrete uses[99] as well as references to propositions of Euclid,[100] marking there too the continuity between scholarly and practical mathematical culture.

Xylander's numerical treatment of Euclid's Elements

Xylander's Euclid corresponds to a German translation from the Greek of the principles and enunciations of the propositions, together with an adaptation of the proofs and a pedagogical commentary, of the first six books of the *Elements*. As Xylander explains in his preface, his aim in producing this translation was to fill a gap in the mathematical knowledge available in German at his time by making Euclid's *Elements* useful and understandable to a German reader who is inexperienced both in geometry and in ancient languages (Greek and Latin).[101] This type of reader, according to Xylander, would include painters, goldsmiths and master builders, and more generally all those who need to count, measure and trace figures by instrumental means.[102] Even if he had written the first draft of his Euclid in 1555 and only started teaching mathematics (at least publicly) in 1562 (i.e. the year of publication of this commentary), Xylander may also have been motivated to have his translation printed for university students, since he had already been teaching at the Faculty of Arts of Heidelberg (lecturing in Greek) since 1558.

Conforming to this pedagogical aim, Xylander often reduced or adapted Euclid's proofs to render them more intelligible and attractive to a lay audience (often addressing the 'simple reader,' or in German 'der einfältige Leser'),[103] notably by using terms from everyday language side by side with the original Greek or Latin terms.[104] He also provided (as Scheubel) many diagrams with compass marks and dotted lines and made the steps of the instrumental construction explicit in many problems of Book I,[105] explain-

99 E.g. Scheubel, *De numeris*, C7r or D5v–D6r.

100 E.g. ibid., B4r–B5r or C7r.

101 Xylander, *Die sechs erste Bůcher Euclidis, An den Leser*, b2r–v.

102 Ibid., b1v: "wie alle künsten […] der Maler, Goldtschmid, Barmaister, &.) sich mit zirckel, lineal, bleywag, ziffern und zalen begehn und behelfen müessen […]."

103 E.g. ibid., *An der Leser*, b1r or I.1, 7.

104 E.g. ibid., Df. I.1, 1: "Ain *punct* oder *tipfflin*" or I.4, 8: "*Basis*, ist Griechisch, heißt der *grund, boden,* oder *fuß* [des triangels]." On this practice, see Angela Axworthy, "Renaissance approaches to the terminology of mathematics," *Le Français Préclassique* 26 (2024): 61–68.

105 E.g. Xylander, *Die sechs erste Bůcher Euclidis,* I.1, 6: "begreiff ich mitt einem zirckel."

ing what is required in practice for the instrumental construction of certain propositions (e.g. the tracing of small intersecting arcs instead of full circles).[106] This was sustained by a reference to geometrical instruments in the title of the book.[107] In some cases, Xylander actually replaced Euclid's construction by a purely mechanical one[108] or suppressed the proof altogether to invite the reader to derive the construction or demonstration from the diagram.[109]

To look at an example of mechanical construction in Xylander's Euclid, let us consider Prop. I.2, which teaches how "To place a straight line equal to a given straight line with one end at a given point."[110] In his treatment of this proposition, Xylander entirely leaves aside Euclid's construction, which involves the tracing of an equilateral triangle and of two circles (as shown in Fig. 5, from Pacioli's 1509 Euclid[111]) to guarantee that, at each step, a line properly equal to the given line is produced until the given point is reached. In Xylander's Euclid (Fig. 6), it is replaced by the simple act of measuring the length of the given line with a compass and then placing one foot of the compass at the given point and marking out the other extremity of the requested line with the other foot of the compass (which has maintained its opening) at any place around the given point, which are then joined by a line equal to the given one.

> II. The Second Proposition. *From a given point, to draw a line which is equal to a given line*. It is done thus.
> Although this may be very easily accomplished, that is, if you take the length of the given line with a compass, and then measure it from the given point and draw a straight line of that extent, which is clear and flawless and founded on the nature of the compass, however those who have written demonstrations for Euclid's propositions have provided here another operation, which is indeed ingenious, but difficult. I have left it aside as unnecessary for the lay German reader (whom I chiefly intend to serve). I trust that I shall not be blamed for it, etc.
> In the figure (Fig. 6), A is the given line, B the point, the lines BC, BD, BE, etc., are all equal to the given line.[112]

106 E.g. ibid., I.1, 7 or I.9, 11.

107 Ibid., title page: *Die sechs erste Bůcher Euclidis, Vom anfang oder grund der Geometri […] (versteh alles kunstlichen, gwisen, und vortailigen gebrauchs des Zirckels, Linials oder Richtscheittes und andrer werckzeüge […]*.

108 This is the case in I.3 (ibid., 7).

109 In I.29 (ibid., 18), the proof is replaced by an indication that the proposition is the converse of the previous proposition and that it can be understood and proven by the figure of the following proposition.

110 Heath, *The Thirteen Books*, vol. 1, 245.

111 Pacioli's commentary on the *Elements* (Pacioli, *Euclidis elementa*) was based on Campanus' Euclid, even if new images and additional comments were added.

112 Xylander, *Die sechs erste Bůcher Euclidis*, I.2, 7.

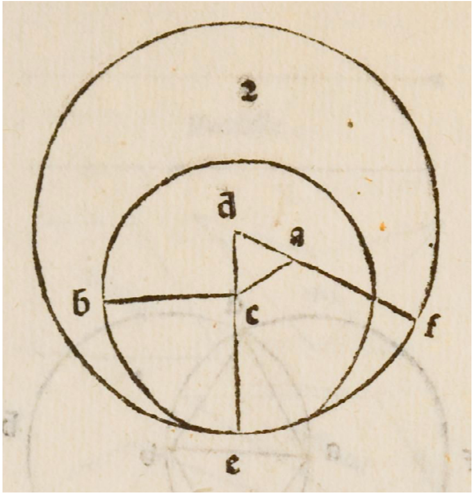


Fig. 5 – Diagram representing the steps of the classical construction of Euclid's Prop. I.2. Pacioli, *Euclidis elementa*, 5v. Image source: Max Planck Institute for the History of Science, Berlin (Public Domain Mark 1.0).

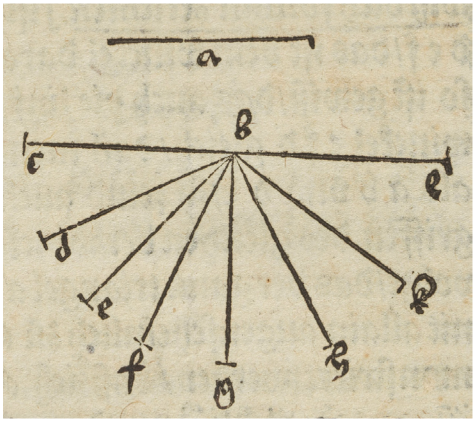


Fig. 6 – Diagram representing the steps of Xylander's construction for Prop. I.2. Xylander, *Die sechs erste Bůcher Euclidis*, Prop. I.2, 7. Image source and permission: Staatsbibliothek Bamberg.

In this framework, the mechanical character of the construction is not only based on the explicit reference to the instrument (i.e. the compass), which contrasts with the absence of reference to instrumental procedures in the *Elements*, but also on the fact of entirely leaving aside the Euclidean construction and demonstration, which are scientifically grounded in the *Elements* through their dependence on Postulates 1-3, Df. I.15 and Prop. I.1. This also contrasts with Scheubel's attitude in Prop. I.3, who had at least presented his mechanical construction *in addition to* the Euclidean classical construction and at least pointed to the principles and propositions on which it depends.

For Xylander, the fact of circumventing Euclid's construction and demonstration, which adds to its mechanical (as opposed to geometrical) character, is then justified by the too great complexity and lack of necessity of Euclid's construction for the readers he addresses, which are unlearned in both mathematics and the languages of ancient texts and of scholars (Greek and Latin). The accompanying diagram (Fig. 6), which depicts the taught construction as having been repeated a certain number of times, in different directions, adds to the operative character and pedagogical aim of Xylander's discourse, which thus invites the reader to perform the procedure several times to learn and master it in practice.

As Scheubel, Xylander also proposed in his commentary an extensive numerical treatment of Euclid's geometrical propositions. But contrary to Scheubel, he added numerical examples in all of the books of the *Elements* on which he commented (I–VI), even if this treatment was not evenly distributed throughout the commentary, being as expected more extensive in Books II, V and VI than in Books I, III and IV.

Xylander, as Scheubel, often placed numerical examples in dedicated commentary sections,[113] which sometimes covered several pages and contained several subsections. In these, he either applied a computational process on different figures (as in Prop. I.41[114]) or taught different arithmetical or algebraic rules (as in Prop. II.4[115]), sometimes together with a presentation of mathematical or practical uses of the proposition.[116] However, contrary to Scheubel, he often included numerical examples in the section that immediately follows the enunciation of propositions,[117] which only loosely corresponds to the proof, in a classical Euclidean sense. In certain cases, Xylander specified that he replaced the geometrical proof of Euclid by a numerical illustration due to its too great obscurity or complexity.[118]

To consider Xylander's numerical approach to Euclid's geometrical propositions, we may take the example of Prop. I.36.[119]

> XXXVI. The Thirty-Sixth Proposition. *All parallelogrammatic figures which stand between two parallel lines and upon equal bases or foundations are equal to one another.*
> *Explanation* (*Erklärung*). This proposition may be easily understood from the previous one, since, between one line and two equal lines, no difference is apprehended and no alteration is produced. Examples follow.
> In the first example, it is also proven by numbers that the rectangle ABCD is equal to the rhomboid EFGH: multiply the sides of the oblong rectangle together, and you have its area. Now, to find the area of the figure EFGH, note that, according to Proposition 34, it is divided by its diagonal FH, which is 6, into two equal triangles; and the sides of each such triangle are known, namely 6, 9, and $\sqrt{117}$. Since the two depicted parallelograms are equal to one another according to this proposition, it necessarily follows that the triangle EFG must contain 27 (that is, $\frac{1}{2}$ of 54).

113 This appears, for example, in the titles of commentary sections, in I.34 (Xylander, *Die sechs erste Bůcher Euclidis*, 21): "*Volgen Exempell, mit ziffern erklert*." or I.47 (ibid., 40): "*Demonstration durch rechnung und zalen*."

114 The various sections of the commentary (ibid., I.41, 27–30) mark the succession of the different figures whose areas are calculated: *Erklårung durch exempel*; *Rechnung der Triangel, vorgesetzter figuren*; *Der dritten Figur; Der vierdten Figur*; *Der fünfften Figur*.

115 Ibid., II.4, 48–54. See *infra*.

116 E.g. ibid., I.41, 30; I.47, 40 or II.1, 45.

117 In theorems, this part is generally called *Erklärung* (i.e. explanation).

118 E.g. *Die sechs erste Bůcher Euclidis*, I.38, 26 or I.41, 27.

119 Heath, *The Thirteen Books*, vol. 1, 331: "Parallelograms which are on equal bases and in the same parallels are equal to one another."

> This may be discovered by the given rule with the aid of art of counting (without which all other arts, but especially those who proceed by compass and straightedge, measure and number, remain defective and imperfect). Sum the sides. From half of the sum subtract each side separately, multiply, etc., as represented above, and you will find the area of the triangle to be 27, as indeed it is. This operation, together with its advantages, I have wished to set down for the reader's sake, exhorting him not to neglect the practice of the art of counting, especially concerning surds and binomials, nor to despise it as useless or too difficult; otherwise the art will often bring him nothing. On this (God willing), more in its proper place.[120]

Instead of providing a geometrical proof in the classical Euclidean style, Xylander started by simply pointing out its evidence on the basis of the previous proposition (I.35), which states that "Parallelograms which are on the same base and in the same parallels are equal to one another"[121] and which is the first proposition in which Xylander adopted a numerical treatment in his commentary on Euclid.[122] He then directly pursues with a numerical example based on the numbers featured in the provided diagram (as shown in Fig. 7), outlining in the text the computational rules to be applied in order to find the areas of the two compared parallelograms ABCD and EFGH situated under the same parallels, that is, the multiplication of the length (9) and width (6) of the rectangular parallelogram ABCD to obtain its area (54) and the computation of the areas of the two triangles with sides 9, and 6 (6 measuring the diagonal of the parallelogram EFGH), following the Heronian method taught in the previous proposition (as Scheubel had done in Prop. I.34). The detailed steps of the second computation are laid out in columns below the text (as shown in the bottom half of the page, in Fig. 7), according to the style of German *Rechenkunst* handbooks. This is the first proposition in which a vertical computations is employed in Xylander's commentary.

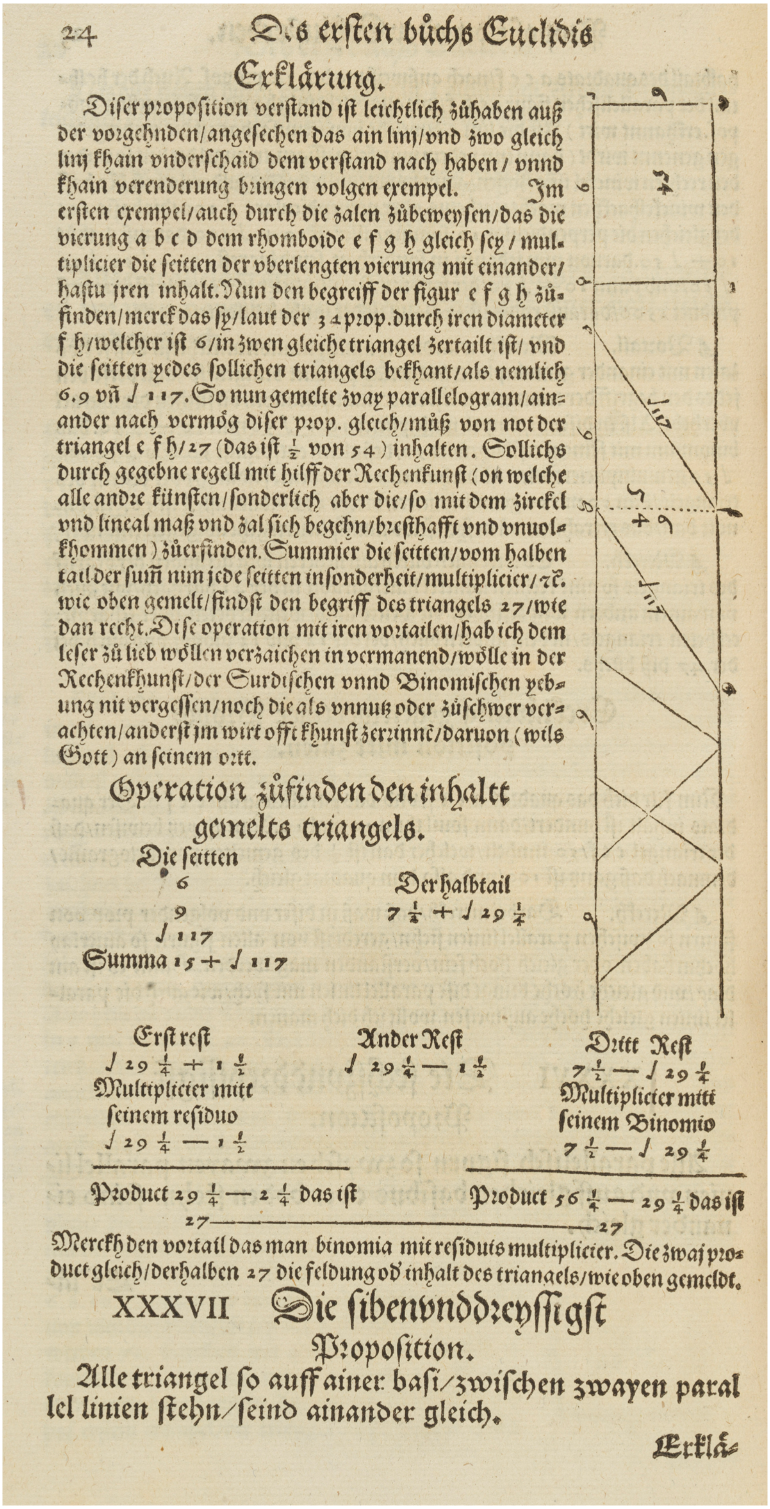
Interestingly, we may note that the computational procedure applied to the numerical example provided by the diagram is justified by its foundation on a prior proposition (Prop. I.34), as was done in classical Euclidean propositions, and in other geometrical propositions (as Prop. I.1), in which Xylander remained more faithful to Euclid.[123] This suggests that, for Xylander, an arithmetical computation, for certain propositions, should

120 Xylander, *Die sechs erste Bůcher Euclidis*, I.36, 24.

121 Heath, *The Thirteen Books*, vol. 1, 326. Xylander, *Die sechs erste Bůcher Euclidis*, I.35, 22.

122 In Prop. I.35, the only geometrical proof that is proposed corresponds to a statement of the given through a general description of the diagram. It is then followed by a long appendix (*Anhang*) teaching the Heronian method of computation of the areas of triangles with specific numerical examples, also in abbreviated form, as in Scheubel's Euclid, to which Xylander added a numerical proof of a converse of the proposition (ibid., 23: *Erklårung gegenwertiger Proposition mitt zalen*).

123 This was the case, for example, in Prop. I.1, ibid., 6–7.



Erklärung.

Diser proposition verstand ist leichtlich zůhaben auß der vorgehnden/angesechen das ain linj/vnd zwo gleich linj khain vnderschaid dem verstand nach haben/vnnd khain verenderung bringen volgen exempel. Im ersten exempel/auch durch die zalen zůbeweysen/das die vierung a b c d dem rhomboide e f g h gleich sey/multiplicier die seitten der vberlengten vierung mit einander/hastu jren inhalt. Nun den begreiff der figur e f g h zůfinden/merck das sy/laut der 34 prop. durch iren diameter f h/welcher ist 6/in zwen gleiche triangel zertailt ist/vnd die seitten yedes sollichen triangels bekhant/als nemlich 6.9 vñ $\sqrt{117}$. So nun gemelte zway parallelogram/ainander nach vermög diser prop. gleich/můß von not der triangel e f h/27 (das ist $\frac{1}{2}$ von 54) inhalten. Sollichs durch gegebne regell mit hilff der Rechenkunst (on welche alle andre künsten/sonderlich aber die/so mit dem zirckel vnd lineal maß vnd zal sich begehn/bresthafft vnd vnuolkhommen) zůerfinden. Summier die seitten/vom halben tail der sum̃ nim jede seitten insonderheit/multiplicier/rc. wie oben gemelt/findst den begriff des triangels 27/wie dan recht. Dise operation mit iren vortailen/hab ich dem leser zů lieb wöllen verzaichen in vermanend/wölle in der Rechenkhunst/der Surdischen vnnd Binomischen yebung nit vergessen/noch die als vnnutz oder zůschwer verachten/anderst jm wirt offt khunst zerrinnẽ/daruon (wils Gott) an seinem ortt.

Operation zůfinden den inhalt gemelts triangels.

Die seitten
6
9
$\sqrt{117}$
Summa $15 + \sqrt{117}$

Der halbtail
$7\frac{1}{2} + \sqrt{29\frac{1}{4}}$

Erst rest
$\sqrt{29\frac{1}{4}} + 1\frac{1}{2}$
Multiplicier mitt seinem residuo
$\sqrt{29\frac{1}{4}} - 1\frac{1}{2}$

Ander Rest
$\sqrt{29\frac{1}{4}} - 1\frac{1}{2}$

Dritt Rest
$7\frac{1}{2} - \sqrt{29\frac{1}{4}}$
Multiplicier mitt seinem Binomio
$7\frac{1}{2} - \sqrt{29\frac{1}{4}}$

Product $29\frac{1}{4} - 2\frac{1}{4}$ das ist 27

Product $56\frac{1}{4} - 29\frac{1}{4}$ das ist 27

Merckh den vortail das man binomia mit residuis multiplicier. Die zwaj product gleich/derhalben 27 die feldung od' inhalt des triangels/wie oben gemeldt.

XXXVII Die sibenvnddreyssigst Proposition.

Alle triangel so auff ainer basi/zwischen zwayen paral lel linien stehn/seind ainander gleich.

Erklä-

Fig. 7 – Xylander, *Die sechs erste Bůcher Euclidis*, Prop. I.36, 24. Image source and permission: Staatsbibliothek Bamberg.

be regarded as having the same degree of validity as a geometrical demonstration, at least for the readership he was aiming.

In his *Explanation* (*Erklärung*), Xylander justified the arithmetical treatment of this theorem, and with this the numerical treatment of magnitudes in general in his commentary on Euclid, by asserting the necessity of arithmetic for geometry, especially in its practical form (i.e. the type of geometry resorting to instruments and measurements), as for all arts in general. With this, Xylander took care to insist on the importance of arithmetical operations on surd numbers and binomials, in spite of their difficulty or apparent uselessness for certain readers. By stating this, Xylander marked the connection between his interpretation of Euclid's *Elements* and the measurement procedures taught in practical geometry treatises, which often involved irrational numbers. This also legitimized the rather frequent and sometimes lengthy explanations of arithmetical and algebraic rules Xylander provided in the commentary on some of Euclid's propositions.

In this regard, in Prop. II.4, which is also practically only treated numerically,[124] with no consideration for the classical Euclidean proof (contrary to how it was treated in Scheubel's Euclid),[125] Xylander provided in the appendix an explanation of the uses and applications of the proposition in the art of counting[126] with various numerical examples. The rules and operations taught in this framework explain 1) how to extract square roots (featuring examples with crossed-out numbers),[127] 2) how to find the next larger or smaller square number,[128] 3) how to find squares roots of binomials,[129] 4) how to add surd numbers,[130] and how to solve the equation corresponding to 'a square + a root = a number' (or in modern terms $ax^2 + bx = c$),[131] which (as was already shown by Scheubel) is founded on the geometrical relation expressed in Prop. II.4. In this framework, Xylander refers to Christoff Rudolff's *Coß*, notably in the section on equations, where the cossic notations

[124] Ibid., II.4, 48–49.

[125] Interestingly, it contains a *Demonstration* section, but this one merely points to an alternative way of considering the dimensioned parts of the diagram (ibid., 49). Scheubel, on the contrary, generally maintained the classical geometrical proof before providing the additional arithmetical or algebraic content.

[126] Ibid., II.4, 50: "*I. Anhang, darinn der nutz und gebrauch diser prop. In der rechenkhunst ettlicher massen erklertt.*"

[127] Xylander, *Die sechs erste Bůcher Euclidis*, II.4, 50: [*in margine*] "Wie man radicem quadratam ainer zal finde."

[128] Ibid., 51: "*II. Anhang das quadrat zůmehren und mindern.*"

[129] Ibid., 52: "*III. Von Binomischen quadraten und wurzeln.*"

[130] Ibid., 53: "*IV. Surdisch addition.*"

[131] Ibid.: "*Demonstration. Der vergleichung in der Coß, oder regel Algebre, da under dreyen quantiteten naturlicher ordnung, die zwo grǒssern der klainern vergleicht werden, als ʒ + ꝝ gleich ainer ledigen zal.*"

for roots and powers derived from this tradition (such as ꝛe, ce, ʒ, J, √) appear.[132] Rudolff's *Coß* was actually the work which Xylander cited the most in his commentary on Euclid.[133] In addition to Rudolff, he also quoted Stifel's *Arithmetica integra,*[134] Scheubel's German edition of Euclid's arithmetical books[135] and the *Rechenbücher* of Peter Apian, Adam Ries and Simon Jacob.[136]

In line with this tradition, Xylander's numerical examples did not only include whole numbers, but also fractions and surd numbers or, as he also called the latter, 'unnatural numbers' (*unnaturliche Zahlen*).[137] Although Xylander, in his commentary on Prop. II.11,[138] wrote that surd numbers are not to be considered as numbers in their own right, at least in the context of the *Elements,*[139] he nevertheless asserted their usefulness for dealing with propositions (such as Prop. II.11) which involve incommensurable magnitudes,[140] providing a numerical example in which he applied a rule derived from Rudolff's *Coß*.[141] This contrasted with the commentators of the *Elements* (such as Campanus and Clavius[142]) who, while proposing a numerical expression of Prop. II.1–10, presented Prop. II.11 as inexpressible through numbers (i.e. through whole numbers) and therefore entirely left aside its arithmetical translation. Xylander's attitude in this regard is comparable to that

132 Ibid.: "Diese Equation oder vergleichung, ist die fünfft bei dem wolgedachten Christoff Rudolffen in seinem Coß bůch."

133 E.g., ibid., I.47, 40: "durch rechnung der Coß, wie das Christoff Růdolff;" II.5: "als du in den exempeln Rudolffs sechen magst" or introd. Book V, 118: "magst sy finden in Christoff Růdolff Coß."

134 Ibid., II.4, 52: "wie dan sollichs der hochgelert herr Michael Stiffel in seiner herrlichen volkhommen Lateinischen Arithmetic fein angezaigt."

135 Ibid., I.41, 30: "Die weil aber der Hochgelert und kunstreich *M. Joann Scheübel, diser künsten professor zů Tübingen, solliches zů end des 9 bůchs Euclidis (das er sambt dem 7 und 8 verteütscht in druckh gegeben*) mit schönen bericht und exemplen gethon, wil ich dem Leser da selbst sollichs zůerholen befelchen, unnd die 42 proposition für die hand nemmen."

136 Ibid., VI.14, 168: "Gib ich dir auß den exemplen, so du inn den Teütschen rechenbüechlin Christoff Rudolffs, Apiani, Adam Risen, Simon Jacobs, unnd andern hast."

137 Ibid., Df. V.3, 121.

138 Heath, *The Thirteen Books*, vol. 1, 402: "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment."

139 Ibid., II.11, 64: "die Surdisch oder Irrational quantiteten kains wegs zalen mögen sein noch genennt werden […] *Euclides selbst* imm zehenden bůch vilmals zůverstehen gibt."

140 Ibid.: "Aber in surdischen, als Residuis und Irrational zalen (also zůsprechen) mag das wolgeschehen."

141 Ibid.: "du dessen in Christoff Růdolffs Coß, bei der sechßten regel das 7 exempel, hast."

142 This is also the case of many later commentators who followed Clavius' commentary, such as Pierre Le Mardelé (1620), André Tacquet (1654), John Leeke and George Serle (1661) or Heinrich Coets (1691).

of Jacques Peletier (1557), who accepted Campanus' statement of the impossibility of expressing Prop. II.11 numerically (again, through whole numbers),[143] while nevertheless teaching how to deal with it by means of irrationals, referring to rules set out in his 1554 treatise on algebra.[144] By contrast with all these authors, Scheubel directly dealt with Prop. II.11 numerically without discussing the status of irrationals. The case of Prop. II.11 thus shows that, despite persisting conceptual limits with respect to the notion of number as defined by Euclid, the introduction of methods drawn from practical arithmetic and cossic algebra in commentaries on the *Elements* contributed to the transformation and expansion of the concept of number in the sixteenth century.

It is noteworthy in this regard that, in Book V, Xylander did not hesitate to express ratios as fractions.[145] Through this treatment, ratios are no longer merely conceived as a relation between quantities but as numbers in their own right, [146] which may themselves be operated on, even if Xylander admitted that it was not proper to Euclid's approach to ratios.[147] This conception is presented as particularly useful for explaining Euclid's Df. V.7,[148] which, together with Df. V.5, defined the equality and inequality of ratios through the concept of equimultiples and raised many difficulties in the early modern period due to its complexity, obscurity and indirectness, given that it relies on comparisons between multiples.[149]

In the context of these definitions, which, in Xylander's Euclid, are placed later in the order of Book V's definitions (most likely due to their complexity[150]), and in which the term 'magnitude' is replaced by that of 'number,'[151] the use of numerical examples enabled

143 Jacques Peletier, *In Euclidis Elementa geometrica demonstationum libri sex* (Lyon: Jean de Tournes and Guillaume Gazeau, 1557), II.11, 60.

144 Peletier, *L'Algèbre* (Lyon: Jean de Tournes, 1554).

145 Xylander, *Die sechs erste Bůcher Euclidis*, Df. V.3, 122.

146 Rommevaux, "Aperçu sur la notion de dénomination;" Malet, "Euclid's swan song."

147 Xylander, *Die sechs erste Bůcher Euclidis*, Df. V.9, 127: "Darbey du dich alwegen der regel und vortailen der bruchrechnung hast zůbehelffen, aber Euclides hatt es nit mögen durch das dividiern beschreiben […]."

148 Ibid.

149 See Df. V.5, in Heath, *The Thirteen Books*, vol. 2, 114: "Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." On the difficulties raised by these two definitions, and by Euclid's theory of ratios and proportions more generally, in the early modern period, see Enrico Giusti, *Euclides Reformatus. La teoria delle proporzioni nella scuola galileiana* (Turin: Bollati Boringhieri, 1993).

150 Dfs. V.5 and 7 correspond respectively to Dfs. V.8 and 9 in Xylander, *Die sechs erste Bůcher Euclidis*, 125–127.

151 Ibid.

Xylander to explain these two definitions more easily by means of operations on numbers. This is visible in the very formulation he used for Df. V.5, in which Euclid's comparison of equimultiples is directly expressed as a demonstration through numerical operations, before being illustrated by different numerical examples with varying equimultiples:

> That four numbers are proportional to one another, and namely, that the first is to the second as the third to the fourth, is demonstrated as follows: if one takes the first and the third an equal number of times (that is, multiplies them by a given number) and afterwards also takes the second and the fourth an equal number of times, then the products of the first and third are either equal to the products of the second and fourth, or they are greater or smaller in the same proportion.[152]

The numerical interpretation of ratios in Book V, through which (as said) ratios are not only taken as relations between whole numbers (e.g. 3 to 4 or 3:4), but also as quotients (e.g. $\frac{3}{4}$),[153] is justified in the commentary on the definition of ratio (Df. V.3), a definition Xylander expanded to include both numbers and magnitudes (*Zahl oder Größe*),[154] by stating that "number underlies all things, and therefore all of their comparisons may, and indeed must, be expressed through numbers."[155] In the commentary on Df. V.4, where Euclid's term 'magnitude' is then rather replaced by the more general term 'quantity' (as in Scheubel's Euclid),[156] Xylander also expanded the objects of Book V to include ratios involving irrational numbers, stating that "although the proportion between these two lines cannot be expressed except by irrational numbers such as 9 and $\sqrt{162}$, they nevertheless bear a proportion to one another."[157]

These examples again confirm that Xylander, although cautious about the status of

[152] Ibid., Df. V.8, 125.

[153] The term 'quotient' appears in the commentary on Df. V.9 (= Df. V.7), ibid., 127: "seind *die quotienten* gleich."

[154] Ibid., Df. V.3, 120: "When two things of the same kind are compared with one another, or estimated, by their number or magnitude (or what will be taken or understood as number and magnitude), such a comparison or confrontation is called a ratio." Cf. Heath, *The Thirteen Books*, vol. 2, 114: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind."

[155] Ibid., Df. V.3, 121: "die zal ann allen dingen hanget, auch derwegen alle deren vergleichungen durch zalen mo̊gen, ja müssen außgesprachen werden."

[156] Ibid., Df. V.4, 123: "Quantities have a proportion with one another when one of them, taken repeatedly or multiplied, can exceed the other." Cf. Heath, *The Thirteen Books*, vol. 2, 114: "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another." On comparable adaptations of Euclid's definition of ratio, see Malet, "Renaissance notions."

[157] Ibid.: "Wiewol diser zwo linien proportion nit mag außgesprochen werden, dann allein mit unnatürlichen zalen, als 9 und $\sqrt{162}$. Dannocht seind sy geproportioniert."

numbers and their relation to magnitudes in the *Elements*, contributed (like Scheubel) to the transformation of the concept of number and ratio from within the Euclidean tradition.

With regard to Xylander's motivations for his numerical approach to Euclid's geometrical propositions, Book V shows that this treatment first had a pedagogical aim, for after asserting that "number underlies all things," he wrote that "anyone can easily understand that in explaining these books and its whole subject, nothing is more convenient than presenting and clarifying examples by means of numbers."[158] For this reason, Xylander made no effort, unlike Scheubel, to provide both geometrical and numerical examples in this framework, as he considered it a useless repetition: "It would often be unnecessary to deal with such things through lines or figures, since they must in any case be tested and explained by numbers."[159]

Within the numerical examples used in his commentary, in which Xylander constantly addresses the reader in the second person as a master would address his pupil[160] (a rhetorical strategy often employed in early modern practical mathematical treatises), the numerical example is also presented as a means for the reader to experience the truth of the proposition first-hand, justifying the fact of leaving aside the classical Euclidean demonstrative reasoning.[161]

In the above-mentioned Prop. I.35, this approach is furthermore presented as useful and recreative: "*An appendix.* If you want to experience the truth and certainty of this proposition and of those that follow through numerical computation (*which is very useful, beside being pleasant and entertaining*), it would be necessary for you to know how to find the area, content or magnitude of each figure, provided that the lengths of the lines that enclose them are known to you."[162]

In this context, the recreativeness of the numerical translation of the geometrical proposition was most likely related to the practical or operative character of this approach, notably marked in Xylander's Euclid by the designation of arithmetical or

[158] Ibid., Df. V.3, 121: "Dieweil nun dem also, kain ain yetlicher leichtlich verstehn, das zůerklåren dise bůcher unnd den gantzen handel nichts bequemer ist, dan das die exempel alle durch zalen fürgeschriben, und erleüttert werden."

[159] Ibid.: "Dann es zum wenigsten unvonnôten, solches durch linien oder figuren zůverrichten, welche doch mit zalen probiert und erklårt müesten werden."

[160] E.g., ibid., I.45, 34: "*Magstu* durch rechnung beweren" or II.4, 50: "wirt *dich* das aussprechen der zalen lehrnen."

[161] E.g., ibid., I.35, 22; I.41, 30 or I.47, 38.

[162] Ibid., I.35, 22: "*Ain anhang.* So du die warhait und gwiß diser und volgender Prop. durch rechnung in zalen erfaren wôlltest (*wellichs seer nutzlich zů dem das es kurtzweilig und lustig ist*) thett nott das du einer yeden figure feldung oder inhallt oder begriff wißtest zefinden, so dir die lenge der linien, darmit sy beschlossen, bekhannt wer."

algebraic rules as 'practices' (*Practick* or *practica*).[163] For in early modern treatises of practical arithmetic or geometry, the fact of performing computations on specific numerical data or manipulating measuring instruments (even if merely in the imagination) to solve hypothetical concrete problems was regularly presented as enjoyable and entertaining, and in some cases as a pleasant, easier, and more meaningful way for the lay reader to understand the theoretical principles of the teaching.[164] The recreativeness of such examples was underlined in works held as belonging to 'mathematical recreations,' as Leon-Battista Alberti's *Ludi matematici*,[165] in which measuring practices and computations represented a means to solve intellectual puzzles, in addition to offering useful solutions to concrete problems.

In Xylander's version of Prop. I.35, the usefulness of the numerical treatment of the proposition is related to its uses for measuring areas, notably in the context of utilitarian activities such as the tiling of the floor of a room, as marked by the example provided in this context.[166] In other propositions, most of all in Books II and V, this numerical approach to geometrical propositions is presented as useful to understand certain rules of practical arithmetic and algebra[167] and also, through the application of the latter to concrete problems, it is presented as useful to the practice of trade, banking or book-keeping.[168]

Unlike Scheubel, Xylander did not however publish a treatise of arithmetic or algebra together with his edition of Euclid and thus referred to the concrete uses of the taught computational procedures (relating to commerce, currency conversions, financial partnerships and measuring practices) directly within his commentary on certain propositions. This is notably the case in the commentary on Prop. V.16,[169] in which several practical examples are brought forward to illustrate the rule of three,[170] which was a fundamental procedure for commercial arithmetic. Hence, the whole commentary on this proposition is committed to the teaching of the rule of three, as is indicated by its subtitle: "Use of this proposition. Foundation and practice of the rule of three."[171] In this framework, Xylander

163 E.g., ibid., II.1, 46; Book V, 118 or V.16, 146.

164 See, for example, Oronce Fine, *Geometria libri duo*, in *Protomathesis* (Paris: Gerard Morrhe and Jean Pierre, 1532), 64r: "practicis geometricarum subtilitatum exercitamentis, novimus plerumque delectari" or Leonard and Thomas Digges, *A geometrical practise, named Pantometria* (London: Henry Bynneman, 1571), A2r: "the reader shall not a little delighte himselfe with the finenesse and subtilitie of their inventions."

165 Kim Williams *et al.*, *The Mathematical Works of Leon Battista Alberti* (Basel: Springer, 2010).

166 Xylander, *Die sechs erste Bůcher Euclidis*, I.35, 22.

167 E.g., ibid., II.1, 46; V.15, 142 or V.16, 143.

168 E.g., ibid., Df. V.3, 122 or VI.4, 158.

169 Ibid., 143. Heath, *The Thirteen Books*, vol. 2, 164: "If four magnitudes be proportional, they will also be proportional alternately."

170 Xylander, *Die sechs erste Bůcher Euclidis*, V.16, 143–146.

171 Ibid., 143: "*Nutz diser proposition. Grund und practick der regel De tri.*"

referred to specific currencies and measurement units (as *Ellen, Pfund, Gulden, Kronen, Heller…*), using the symbols commonly employed in *Rechenbücher,* as fl. and ß to represent Guldens (or Florins) and Schillings.[172]

Xylander's Euclid thus came closer, in its style (in addition to its language), to Scheubel's 1555 edition of Euclid's arithmetical books and to vernacular handbooks of practical mathematics than to Scheubel's Latin Euclid.[173] It is notable in this respect that Xylander's commentary on Euclid was quoted by Jan Pieterszoon Dou[174] as one of the mains sources of his Dutch translation of Euclid, first published in 1606,[175] along with Dou's own treatises of practical geometry (*Practijck des Lantmetens* and *Van het gebruyck der Geometrische instrumenten*), which he wrote with the surveyor, cartographer and engineer Johann Sems.[176]

The practical character of Scheubel's and Xylander's numerical treatment of Euclid

The numerical approach to magnitudes proposed in Scheubel's and Xylander's commentaries on Euclid conferred a practical character on these works, whereby they diverged from classical expositions of the *Elements*. Through their teaching of rules for the computation of the areas of figures, they related to practical geometry treatises that dealt with the measurement of surfaces, notably in concrete situations (e.g. the measurement of fields or the division of estates).[177] They also related to handbooks of practical arithmetic and cossic algebra through their inclusion of various arithmetical and algebraic rules (particularly in the tradition of Rudolff's *Coß*) in the commentary on certain propositions, such as Prop. II.4. Their link to these traditions is marked in particular by the references made by Scheubel and especially Xylander to the usefulness of Euclid's geometry for practitioners such as surveyors and merchants.

In early modern treatises of practical mathematics, the practical character of the teaching resides not only in its attention to the concrete uses of geometry and arithmetic, but also in its focus on the operations (computational or instrumental) required to perform measurements or constructive procedures, as opposed to the demonstration of their sci-

172 Ibid., 145–146.

173 It is interesting to note that, even if Xylander's Euclid was published after Scheubel's two printed editions of Euclid, Scheubel owned a copy of this work. See Hughes, "The Private Library," 422: "Xylander in sex priores libros Euclidis, teutsch."

174 Jan Pieterszoon Dou, *De ses eerste Boucken Euclidis, Van de beginselen ende fondamenten der Geometri* (Leiden: Orlers, 1607), A4r–v.

175 Dou's other source was the French translation of Euclid by the architect and engineer Jean Errard de Bar-le-Duc (1598). Ibid., A4r.

176 Ibid., A2r–v.

177 Scheubel's library actually counted practical geometry treatises. Hughes, "The Private Library."

entific validity (let alone the consideration of the properties of the measured or constructed objects). Now, it is primarily in this operative sense that Scheubel and Xylander used the expressions *Numerorum praxis* and *Practick* to designate commentary sections that teach computational rules or propose a numerical translation of Euclid's propositions, these sections teaching the different steps of the computational processes required for the determination of the areas of given or sought magnitudes.

The practical character of Scheubel's and Xylander's approach to Euclid's geometry is also reflected in their use of a multitude of different numerical examples to teach arithmetical or algebraic rules or to express certain propositions numerically. The application of a given computational procedure to multiple examples, involving different values and types of quantities (integers, fractions, irrationals, binomials), would allow the reader to empirically grasp its mode of operation and memorise its different steps through repetition, verifying all the while its validity for all the cases considered, as was often done in abbacus handbooks and German *Rechenbücher*.[178] When used to 'demonstrate' a Euclidean proposition numerically, as Xylander chose to do for Prop. I.41,[179] multiple numerical examples would thus enable one to inductively derive its truth or validity from the repeated confirmation that the stated quantitative relation (e.g. the equality of any triangle to half of the parallelogram set on the same base and between the same parallels in Prop. I.41) holds for any of the proposed cases.

The empirical character of the numerical translation of Euclidean definitions or propositions was actually acknowledged by Niccolò Tartaglia, a mathematician and engineer who taught in a *scuola d'abbaco* and who published in 1543 the first vernacular edition of the *Elements*,[180] primarily addressing (like Xylander) a lay audience.[181] As he observed in his commentary on Book V, "often the student who sees the provided proposition verified by the experience of numbers does not trouble himself to understand it by demonstration."[182] Although Tartaglia then intended to warn the reader that reliance on numerical examples may lead to errors if not sustained by geometrical reasoning,[183] he acknowledged thereby the value of numerical examples to facilitate the comprehension of propositions

178 Høyrup, *The World of the Abbaco*.

179 Xylander, *Die sechs erste Bůcher Euclidis*, 27–30. See *supra*, n. 114, 118 and 161.

180 Tartaglia, *Euclide*. On Tartaglia's life and work, see Arnoldo Masotti, "Tartaglia (also Tartalea or Tartaia) Niccolò," *Dictionary of Scientific Biography*, vol. 13, ed. Charles Coulston Gillispie (New York: Charles Scribner's Sons, 1976), 258–262.

181 The fact that vernacular translations of Euclid such as those of Tartaglia and Xylander were addressed to early students, laymen or practitioners is indicated in the title of their works. For Tartaglia, see *Euclide*, title page: "*ogni mediocre ingegno, senza la notitia, over suffragio di alcun'altra scientia con facilita, sera capace à poterlo intendere*."

182 Tartaglia, *Euclide*, Df. V.11, 64v: "spesse volte il studente che vede con la esperientia de numeri verificarse la propositione preposta, non si cura di intendere quella per demostratione."

183 Ibid., Df. V.11, 64v.

often too complex for beginners, a benefit acknowledged by later commentators of Euclid, as Didier (or Denis) Henrion, who defended numerical examples as means of abbreviating lengthy and obscure demonstrations.[184] Now, in Tartaglia's commentary on Euclid, the use of numerical examples, even accompanied by geometrical demonstrations, was explicitly associated with the approach of practitioners.[185]

Yet, despite their connections with treatises of practical mathematics, Scheubel's and Xylander's commentaries on Euclid represented expositions of the *Elements* in their own right. This is evident in their general observance of the list and order of Euclid's principles and propositions found in the classical editions of the *Elements*, even if minor changes were sometimes introduced for pedagogical purposes.[186] The Euclidean origin of the principles and enunciations of propositions, although often reformulated or expanded by Xylander by the addition of synonyms or complementary information to facilitate their comprehension, remains clearly recognisable. In Scheubel's case, the classical geometrical proofs are generally maintained, though sometimes abbreviated or supplemented by more mechanical or empirical reasonings, as in Prop. I.3. Similar tendencies appear, though less frequently, in Xylander.[187] Xylander's and Scheubel's commentaries on Euclid, therefore, cannot be reduced to treatises of practical geometry or to *Rechenbücher*, whose traditions, contrary to that of the *Elements*, were not built around a unique and common source.[188] Nevertheless, these works show how, in the sixteenth century, certain commentaries on Euclid's brought together features traditionally associated with both theoretical and practical mathematical traditions.[189]

Scheubel and Xylander's Euclid and the Protestant universities of Southwest Germany

What does Scheubel and Xylander's approach to Euclid's *Elements* say about the type of mathematical teaching and practice that was promoted in the cultural and institutional context in which they were active? Indeed, if Scheubel and Xylander present compara-

184 Didier (or Denis) Henrion, *Les quinze livres des Elements geometriques d'Euclide* (Paris: Isaac Dedin, 1632), II.1, 83.

185 Tartaglia, *Euclide*, X.47, 154r–v: "*da prattici* se descrivera in questa forma 12 piu 63."

186 E.g. the reduction of the postulates to Euclid's first three postulates (in both Scheubel and Xylander) or the change in the order of the definitions of Book V in Xylander. The former was often found in other editions of Euclid published at the time.

187 Propositions that remain overall conform in their structure to the classical Euclidean proofs, such as Prop. I.1, were not considered here as they are less relevant to the object of this study.

188 This was already made clear by Hugh of St Victor at the beginning of his twelfth-century *Practica geometriae*. Frederick A. Homann, *Hugh of St Victor. Practical geometry = Practica geometriae* (Milwaukee: Marquette Univ. Press, 1991), 33.

189 On this phenomenon, see Axworthy, "The Hybridization."

ble treatments of Euclid's *Elements*, these authors are also comparable in their geographical and cultural origins, as well as in their professional and institutional status. They were indeed both born in the southwestern region of present-day Germany (Scheubel in Kirchheim unter Teck and Xylander in Augsburg) and both ultimately held a professorship in mathematics at a university situated relatively near their birth places.[190] Furthermore, at the time, each of these institutions adopted comparable pedagogical, institutional and religious principles. For although the University of Heidelberg was founded by Pope Urban VI prior to the Reformation and was initially hesitant to adopt humanist culture and pedagogical practices, unlike the University of Tübingen, established in 1477 by promoters of Humanism,[191] both institutions had, by the mid-sixteenth century, become active centres for the development of humanist learning and for the promotion of Protestantism.[192]

As was asserted by Paul Grendler, there was a strong connection between the birth and spread of Protestantism and the humanist intellectual life of reformed German universities.[193] The University of Heidelberg, though oscillating at the time between Lutheran and Calvinist doctrines, was the second most important university for the Protestant Reformation after Martin Luther's University of Wittenberg. And the University of Tübingen, where both Scheubel and Xylander studied and/or taught, as did many renowned sixteenth- and seventeenth-century German mathematicians such as Johannes Stöffler, Michael Mästlin, Johannes Kepler and Wilhelm Schickard, also played a considerable role in this regard.[194] Gerard Betsch has, for that matter, noted the frequent correlation between the interest in mathematics (and especially in practical mathematics) from these Tübingen students or professors and their involvement in

[190] There is about 40 km between Kircheim and Tübingen and about 250 km between Augsburg and Heidelberg.

[191] I.e. Count Eberhard im Bart (later first Duke of Württemberg) and his mother Mechthild von der Pfalz.

[192] James H. Overfield, *Humanism and Scholasticism in Late Medieval Germany* (Princeton: Princeton University Press, 1984); Thomas A. Howard, *Protestant Theology and the Making of the Modern German University* (Oxford: Oxford University Press, 2006), Chap. 2; Ulrich Köpf, *Die Universität Tübingen und ihre Theologen* (Tübingen: Mohr Siebeck, 2020).

[193] Paul Grendler, "The Universities of the Renaissance and Reformation," *Renaissance Quarterly* 57, 1 (2004): 1–42.

[194] Gerard Betsch, "Praxis geometrica und Kartographie and der Universität Tübingen um 16. und frühen 17. Jahrhundert," in *Zum 400. Geburtstag von Wilhelm Schickard: zweites Tübinger Schickard-Symposium*, ed. by Friedrich Seck (Thorbecke: Sigmaringen, 1995), 185–226; Id., "Südwestdeutsche Mathematici aus dem Kreis um Michael Mästlin," in *Der „mathematicus": zur Entwicklung und Bedeutung einer neuen Berufsgruppe in der Zeit Gerhard Mercators*, ed. by Irmgard Hantsche (Bochum: Brockmeyer, 1996), 121–150.

religious offices.[195] This could be partly explained by the predominant importance and interrelation (demonstrated again by Grendler[196]) of the Faculty of Arts and the Faculty of Theology in Protestant universities in the sixteenth century, as well as by the important social cohesion that existed in this framework between students and professors, as between the universities and the local communities. For in reformed German universities, unlike Italian universities, not only were professors living among students, which fostered a form of solidarity between the former and the latter, but German students most often only aimed to obtain a bachelor, in order to teach in local schools or work in local administrations, which contributed to bind the needs of the university with those of the city. This situation did not so much exist in Italian universities, at least in the first half of the sixteenth century, where intellectual competition was much stronger and where the various faculties, their members, and the local communities, did not share the same social connection. Also, contrary to the students of reformed German universities, a bachelor's degree was generally not offered in Italian universities. These rather prepared students for the doctorate (especially in law and medicine), or at least for the Master's degree.

In this regard, Scheubel's biographers surmised that it is very likely due to his Lutheran convictions that he left the University of Leipzig to pursue his studies in Tübingen in 1535,[197] as the Duchy of Württemberg (and with it the University of Tübingen[198]) had officially turned to Protestantism in 1534, while Leipzig did so only in 1539. It should be noted for that matter that the paratext of his *De numeris et diversis rationibus seu regulis computationum opusculum* from 1545 counts a short versified poem by Philipp Melanchton,[199] who played a key role in the reformation of the University of Tübingen.[200]

Xylander, who was himself in contact with Melanchton, Ulrich Zwingli and Thomas Erastus (among other important Protestant thinkers), studied philosophy at Tübingen under the Lutheran Aristotelian philosopher Jakob Schegk.[201] Furthermore, Xylander held in 1564 the function of secretary for the assemblies called at the monastery of Maulbronn to discuss the various points debated by the Lutheran and Calvinist doctrines and again in 1571 at the colloquium with the Anabaptists in Frankenthal. It may also be noted that the Protestant Parisian royal lecturer in mathematics Petrus Ramus

195 Betsch, "Südwestdeutsche Mathematici."
196 Grendler, "The Universities."
197 Staigmüller, "Johannes Scheubel," 435; Day, *Scheubel as an Algebraist*, 15; Reich, *500 Jahre Johann Scheubel*, 66.
198 Richard L. Harrison Jr., "Melanchthon's Role in the Reformation of the University of Tübingen," *Church History* 47/3 (1978): 270–278.
199 Scheubel, *De numeris*, i6v.
200 Harrison, "Melanchthon's Role."
201 Gall, "Xylander" and Schöll, "Xylander."

recommended Xylander for the chair of mathematics at Heidelberg in 1561.

Moreover, even if Xylander had started working on his Euclid in 1555, and thus before he started teaching at the University of Heidelberg, it was finalised and published during his lectureship at this institution. As a former student of the University of Tübingen, who possibly studied under Scheubel in 1549–1550, he may have been inspired to undertake this translation of the *Elements* by the latter's lectures, as by the publication of his German edition of Euclid's arithmetical books in 1555, in addition to the desire to support the artisans, merchants and bankers of Augsburg, among other German cities.

Scheubel and Xylander therefore fully instantiate the intertwinement proper to the universities of Southwest Germany between the adoption of Protestantism, the promotion of humanist culture, and the efforts to cater to the needs of the local communities. Their interest in practical and applied mathematical knowledge, notably in the practices of German *Rechenmeister*, is coherent with the observations made by Pietro D. Omodeo concerning the mathematical curricula of the Protestant University of Helmstedt, where an important place was attributed to practical mathematical works, in particular the practical arithmetic of the Dutch professor of mathematics Gemma Frisius, which was published in Wittenberg, among other places.[202] We can also find a similar orientation in Johannes Sthen's pedagogically-oriented adaptation of Book VII of the *Elements* published in Wittenberg in 1564, which is presented as offering a foundation for the teaching of the rules of practical arithmetic.[203] Even if it only proposes a very limited and superficial numerical treatment of the *Elements* in comparison with the editions of Scheubel and Xylander, it is one of the few sixteenth-century commentary on Euclid that compares to these in its explicit desire to connect the content of the *Elements* with the practices of common arithmetic.

To this must be added that Scheubel and Xylander's approach to Euclidean geometry is also coherent with the pedagogical model promoted by Ramus for mathematics, since the latter defended a definition of arithmetic and geometry as the arts of counting and measuring well (*doctrina bene numerandi* and *ars bene metiendi*)[204] and transmitted theoretical principles of Euclidean geometry by combining them with a teaching on the

202 Pietro D. Omodeo, "The German and European network of the professors of mathematics at Helmstedt in the sixteenth century," in *The Circulation of Science and Technology: Proceedings of the 4th International Conference of the ESHS*, ed. by Antoni Roca-Rosell (Barcelona: Societat Catalana d'Història de la Ciència i de la Tècnica, 2012), 294–301.

203 Johannes Sthen, *Arithmetices Euclideae*, title page: "*vera principia ac solidiora fundamenta Logistices, id est, ut vocant, Arithmetices Practicae.*"

204 Petrus Ramus, *Arithmeticae libri duo; Geometriae septem et viginti* (Paris: André Wechel, 1569). It should be noted that this book was part of Scheubel's library. Hughes, "The Private Library," 422: "Petri Ramus arithmetica et geometria."

procedures of practical geometry.[205] It is interesting to note for that matter that Ramus' *Algebra* (1560),[206] which was taught and further developed by professors of German Protestant schools that followed Ramist pedagogical methods in the 1580s, drew much from Scheubel's *Algebrae descriptio*.[207]

Hence, the editions of Euclid's *Elements* by Scheubel and Xylander may be regarded as instantiating an approach to mathematics that was representative of sixteenth-century Protestant pedagogical reforms. This approach is not only defined by the fact of giving an important place to practical and applied mathematics (commercial arithmetic, surveying, instrument-making and the like) in the Arts curriculum, but also by the fact of mixing the geometrical principles and propositions of Euclid with the procedures of practical mathematics, making them thereby more accessible, meaningful and useful to students, conceived as members of their social community.

Conclusion

Scheubel and Xylander's numerical approach to the *Elements* holds a unique place in the sixteenth-century Euclidean tradition by the strong connection it made between Euclid's geometry and the practice of German *Rechenmeister*. Nevertheless, their editions of Euclid remained the works of scholars, who studied and taught at the university and were versed in ancient Greek and Latin sources. In their work, one thus finds a pedagogical model defended by Protestant humanists, which aimed to make ancient texts both accessible and useful to their readers, as members of their community.

Such an approach to Euclid would become much more frequent in the seventeenth century, notably among editions of the *Elements* published in Northern Europe, in areas which had adopted the doctrine of the Reformed Church, such as those of Lucas Brunn, Tobias Beutel, Heinrich Meißner in Germany,[208] but also Christoff Dybvad in Denmark,[209] Martin

205 Menghini, "From Practical Geometry."

206 Petrus Ramus, *Algebra* (Paris: André Wechel, 1560).

207 François Loget, "De l'algèbre comme art à l'algèbre pour l'enseignement: Les manuels de Pierre de La Ramée, Bernard Salignac et Lazare Schoner," *Revue de Synthèse* 132/4 (2011): 495–527. On Scheubel's influence on sixteenth-century algebra, see also Day, *Scheubel as an Algebraist*, 19–24 and Reich, *500 Jahre Johann Scheubel*, 79–82.

208 Lucas Brunn, *Euclidis Elementa Practica, Oder Außzug aller Problematum und Handarbeiten auß den 15. Büchern Euclidis* (Nürnberg: Simon Halbmayer, 1625); Tobias Beutel, *Geometrischer Lust-Garten. Darinnen die edele und höchstnützliche schöne Kunst Geometria, Aus den Euclide gepflantzet* (Leipzig: Christian Michael, 1660); Heinrich Meißner, *Des Gantzen, In 15 Büchern bestehenden, Teutschen Euclidis* (Hamburg: Wieringen, 1690).

209 Christoffer Dybvad, *In Geometriam Euclidis prioribus sex Elementorum libris comprehensam Demonstratio Numeralis* (Arnhem; Leiden: Christopher Guyot, 1603).

Gestrinius in Sweden,[210] Frans Van Schooten in the Dutch Republic[211] or William Alingham in England,[212] who proposed arithmetical and algebraic treatments of the geometrical books of the *Elements* combined with a practical approach to Euclid's propositions.[213] In comparison, very few seventeenth-century Italian commentators of the *Elements* followed a similar approach (with the exception of Pietro Antonio Cataldi[214]) in spite of the great number of different editions published in Italy at this time.[215] This also concerns the expositions of the *Elements* of two disciples of Galileo, namely Giovanni Alfonso Borelli and Vincenzo Viviani, who maintained in this respect a rather theoretical approach to geometry, in spite of their attempts to revise or transform Euclid's *Elements*, particularly the theory of ratios contained in Book V, to make it more applicable to the apprehension of the physical world, in line with Galileo's own project of revision of Euclidean geometry.[216]

To which extent the above-mentioned numerical Northern European editions of the *Elements* were determined by the Protestant background of their authors and may be placed in continuity with the editions of Scheubel and Xylander in this respect, as with regard to the specificities of their numerical approach to Euclid's geometrical books, would require a separate inquiry, particularly because many other factors need to be considered for the numerical treatment of Euclid's *Elements* in this period, such as the emergence of modern algebra and analytical geometry, together with the greater focus placed by mathematicians on problem-solving techniques (over the demonstration of theorems), the in-

210 Martin Gestrinius, *In geometriam Euclidis demonstrationum libri sex* (Uppsala: Aeschillus Matthiae, 1637). On Gestrinius' algebraic approach to Euclid, see Johanna Pejlare and Staffan Rodhe, "On the relations between geometry and algebra in Gestrinius' edition of Euclid's Elements," in *History and Pedagogy of Mathematics. Proceedings of the 2016 ICME Satellite Meeting* (Montpellier: IREM de Montpellier, 2016), ed. by Luis Radford *et al.*, 513–523.

211 Frans Van Schooten, *De propositien van de XV boucken der elementen Euclidis demonstratis* (Leiden: Govert Basson, 1617).

212 William Alingham, *Geometry epitomiz'd: being a compendious collection of the most useful propositions in the first, third, fourth, fifth and sixth books of Euclid. Together with their uses, in several practical parts of the mathematicks. Also, Euclid's second book and doctrine of proportion algebraically demonstrated. With some of the most useful problems required in practise* (London: J. Moxon and B. Beardwell, 1684).

213 It is unfortunately not possible to propose an analysis of these sources in the framework of this study. An examination of this later tradition is forthcoming.

214 Pietro Antonio Cataldi, *I primi sei libri de gl'Elementi d'Euclide ridotti alla Prattica, dove si mostrano le Inventioni delle Regole Geometriche, & Algebratiche necessarie, & di continuo uso* (Bologna: Sebastiano Bonomi, 1620) [First ed. 1613].

215 There were more than twenty different editions of Euclid published in Italy during the seventeenth century.

216 Vincenzo De Risi, "Euclid upturned: Borelli on the foundations of geometry," *Physis* 57/2 (2022): 1–23; Angela Axworthy, "Mathematics in the Accademia del Cimento: from a language of Nature to a language of Reason," *Physis* 59/2 (2024): 463–499.

tensification of mercantilism, the changing place and status held by practical mathematics in mathematical culture and university curricula, together with the expansion of mathematical instruction and numeracy beyond the university, as well as the greater ability of printers to cater to new mathematical languages and notations. It remains that the editions of the *Elements* by Scheubel and Xylander may be considered as having inaugurated a phenomenon that would become crucial to the reshaping of the Euclid's treatise, and thereby to the transformation of elementary geometry in early modern Europe, contributing to bring together arithmetic and geometry as well as theoretical and practical mathematics.

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