# The influence of practical arithmetic on the emergence of proto-symbolic algebra: A long-run survey

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**Abstract**
The paper provides a long-run overview of how the development of what we may term "proto-symbolic algebra" was profoundly shaped by the tradition of practical arithmetic, particularly as it evolved through abacus mathematics. Within this milieu, algebraic problem-solving techniques became increasingly sophisticated, and symbolic representations began to emerge. While the recovery of classical mathematical works provided an ideal of mathematical generality and abstraction, the mathematics practiced in abacus schools supplied computational techniques that were essential for the operationalization of algebra. Key figures such as Tartaglia, Cardano and Bombelli displayed influences of abacus mathematics in their algebraic works. The persistence of practical arithmetic techniques within these works suggests that the evolution of algebra was not a linear progression from classical to modern mathematics but rather a complex synthesis of diverse traditions.

**Keywords**
proto-symbolic algebra, abacus mathematics, hybridisation

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A possible starting point for the historiography on the emergence of new algebraic techniques in the European late Middle Ages can be identified in Pietro Cossali's *Origine, trasporto in Italia, primi progressi in essa dell'Algebra*, published in 1797–99.[1] Cossali was the first to identify Leonardo Pisano's *Liber abaci* as an important early source. Cossali correctly recognised the *Liber abaci* as an early thirteenth-century text, and was among the first to read its algebraic contents from extant manuscripts. Before Cossali's work, the history of algebra had been reconstructed solely from printed sources. The earliest and most famous of these was Luca Pacioli's *Summa de Arithmetica Geometria Proportioni et Proportionalità*, published in Venice in 1494, which Cossali used as a key comparison between Leonardo Pisano's and late fifteenth-century algebra.

Even though he was driven by an understanding of the evolution of mathematical thought that not all historians of mathematics would today subscribe to, Cossali correctly attributed the origins of these new algebraic techniques to Arabic mathematics, particularly al-Khwārizmī's *Al-jabr*, even though he did not know the existing Latin translations of that work. Nonetheless, his hypothesis that intermediary algebraic treatises must have existed between Fibonacci and Pacioli shaped historians' understanding for more than two centuries, despite the limitations of his evidence.

An important contribution was made by Guglielmo Libri's *Histoire des Sciences Mathématiques en Italie* (1838–1841).[2] Libri confirmed that a new form of algebra entered Europe through Latin translations of al-Khwārizmī, and published the text by Gerardo da Cremona. He noted that he found no evidence of algebraic texts in the thirteenth century beyond Leonardo Pisano's works, and reproduced in print the fifteenth chapter of his *Liber abaci*. Moreover, Libri identified a number of abbacists from the fourteenth century, including Paolo dell'Abaco, Paolo Gerardi, and Giovanni Danti, as early transmitters of algebraic ideas, and mentioned having read a number of anonymous manuscripts. He noted that some of these abacus manuals discussed equations of third and higher degrees, but he described some of their algebraic formulas as "des règles bizarres fondées sur de faux principes."[3]

Further progress was made by Baldassarre Boncompagni, who published the surviving works of Fibonacci and described numerous fourteenth- and fifteenth-century Italian manuscripts containing algebraic material.[4] The editions sponsored by Boncompagni pro-

1 Cossali, *Origine, trasporto in Italia, primi progressi in essa dell'algebra* (Parma: 1797). See also Franci, "Pietro Cossali storico dell'algebra," in *Pietro Riccardi (1828-1898) e la storiografia delle matematiche in Italia*, ed. by Barbieri, Cattelani (Modena: Università degli Studi di Modena, 1989), 199–217.

2 Libri, *Histoire des sciences mathématiques en Italie*, 4 vols. (Paris: Jules Renouard, 1838–1841).

3 Ibid., vol. 2, 212–213.

4 Boncompagni, *Intorno ad alcune opere di Leonardo Pisano matematico del secolo decimoterzo* (Roma: Tipografia delle Belle Arti, 1854); Id. (ed.), *Scritti di Leonardo Pisano. Vol. I., Il Liber*

vided an important groundwork for later scholarship, and, for some of Fibonacci's works, they are still today the only modern editions available.[5] It was not until the mid-twentieth century that Gino Arrighi resumed this line of research, transcribing and studying abacus treatises more broadly. In 1980, Warren Van Egmond published his seminal *Catalogue of Italian Abbacus Manuscripts*, which provided for the first time a nearly comprehensive listing of known abacus manuals, several of which include algebraic sections.[6]

Building on this evidence, Laura Toti Rigatelli and Raffaella Franci initiated a series of studies on abacus treatises from the 1980s. Using Van Egmond's catalogue, they and their students transcribed and analysed several fourteenth- and fifteenth-century abacus manuals, many of which were published in the series *Quaderni del Centro Studi di Matematica Medioevale* of the University of Siena.[7] By surveying and providing access to these primary sources, their research contributed to transforming the historiography of medieval Italian algebraic techniques from a largely speculative narrative into a documentary-based literature.[8]

*Abbaci, Vol. II. La Practica Geometriae, Vol. III Opuscoli*, III vol. (Roma: Tipografia delle scienze matematiche e fisiche, 1857); Id., "Intorno ad un trattato d'aritmetica stampato nel 1478," *Atti dell'Accademia Pontificia de' Nuovi Lincei* 16 (1862). See also Woepcke, *Sur l'introduction de l'arithmétique indienne en Occident et sur deux documents importants publiés par le prince don Balthasar Boncompagni et relatifs à ce point de l'histoire des sciences.* (Rome: Imprimerie des sciences mathématiques et physiques, 1859).

5 For a recent critical edition of the *Liber abaci*, see Giusti and D'Alessandro (eds.), *Leonardi Bigolli Pisani vulgo Fibonacci Liber Abbaci* (Firenze: Olschki, 2020).

6 Van Egmond, *Practical Mathematics in the Italian Renaissance: a Catalog of Italian Abbacus Manuscripts and Printed Books to 1600* (Firenze: Giunti Barbera, 1980).

7 See, among others, Salomone, ed., *La reghola de algebra amuchabale: dal codice L.4.21 della Biblioteca comunale di Siena* (Siena: Università degli studi di Siena, 1982); Franci, ed., *Questioni d'algebra: dal Codice L.IX.28 della Biblioteca Comunale di Siena* (Siena: Università degli studi di Siena, 1983); Pieraccini, ed., *Chasi exenplari alla regola dell'algibra: nella trascelta a cura di m. Benedetto: dal codice L.VI.21 della Biblioteca comunale di Siena* (Siena: Università degli studi di Siena, 1983); Procissi, ed., *Ragionamenti d'algebra: i problemi. Dal codice Pal. 567 della Biblioteca nazionale di Firenze* (Siena: Università degli studi di Siena, 1983); Gori and Franci, eds., *Libro di ragioni e misure in sunto e a mente: dal Codice L.IX.30 della Biblioteca comunale di Siena* (Siena: Università degli studi di Siena, 1984); Franci and Pancanti, eds., *Il trattato d'algibra: dal manoscritto Fond. prin. 2. 5. 152 della Biblioteca nazionale di Firenze* (Siena: Università degli studi di Siena, 1988); Simi, ed., *Regole di geometria e della cosa: Codice Palatino 575 (sec. 15.) della Biblioteca Nazionale di Firenze* (Siena: Università degli studi di Siena, 1992); Franci, ed., *Maestro Dardi (s. xiv) Aliabraa Argibra (ms. I.VII.17 Biblioteca Comunale di Siena)* (Siena: Università degli studi di Siena, 2001), vol. xxvi.

8 See, among others, Franci and Rigatelli, "Maestro Benedetto da Firenze e la storia dell'algebra," *Historia Mathematica* 10/3 (1983): 297–317; Id., "Towards a history of algebra from Leonardo of Pisa to Luca Pacioli," *Janus* 72 (1985): 17–82; Iid., "Fourteenth century Italian algebra," in *Mathematics from manuscript to print, 1300-1600*, ed. by Hay (Oxford: Clarendon Press, 1988),

More recent studies, such as those by Albrecht Heeffer, Jens Høyrup, and Van Egmond himself, have contributed to highlight both the characteristics of this mathematical tradition and its internal evolution, bringing to the fore the ways in which it both laid the groundwork for what would eventually become European symbolic algebra, and how it was characterised by its own distinctive features.[9] Heeffer, for example, has suggested that we could understand the algebraic techniques that emerged in abacus treatises as a form of "proto-symbolic" algebra, representing an important transitional stage between purely rhetorical methods and the emergence of symbolic notations.[10]

Scholars such as Cynthia Hay, Barbara Gärtner, Sabine Rommevaux, Maryvonne Spiesser, and Maria Rosa Massa Esteve, meanwhile, have studied mathematical works published beyond the Alps that also contributed to the emergence of new forms of algebra. Rather than a coherent body of mathematical knowledge, sixteenth-century algebra appears as a "plurality" of mathematical techniques characterised by a high degree of experimentalism.[11] In the meanwhile, new historiographical approaches proposed by Karine Chemla and others have stressed the historicity of mathematical knowledge, and the need to understand its evolution as embedded in its historical environment.[12] Rather than a

11–29; Franci, "Leonardo Pisano e la trattatistica dell'abaco in Italia nei secoli XIV e XV," *Bollettino di storia delle scienze matematiche* 23/2 (2003): 33–54; Id., "The History of algebra in Italy in the 14th and 15th centuries: some remarks on recent historiography," *Actes d'història de la ciència i de la tècnica* 3/2 (2010): 175–194; Id., "Il trattato d'arismetricha (Ms. Ricc. 2252 della Biblioteca Riccardiana di Firenze)," *Bollettino di Storia delle Scienze Matematiche* 38/1 (2018): 93–126.

9 Høyrup, "Jacopo da Firenze and the beginning of Italian vernacular algebra," *Historia Mathematica* 33/1 (2006): 4–42; Van Egmond, "The Study of Higher-Order Equations in Italy before Pacioli," *Acta historica Leopoldina* 54 (2008): 303–320; Heeffer, "On the Nature and Origin of Algebraic Symbolism," in *New Perspectives on Mathematical Practices: Essays in Philosophy and History of Mathematics*, ed. by van Kerkhove (Singapore; Hackensack, NJ: World Scientific, 2009), 1–27; Id., "The Genesis of the Algebra Textbook: From Pacioli to Euler," *Almagest* 3/1 (2012): 26–61; Høyrup, *Explorations and False Trails: The Innovative Techniques That Brought About Modern Algebra* (Cham: Springer, 2024); Id., *The World of the Abbaco: Abbacus Mathematics Analyzed and Situated Historically between Fibonacci and Stifel* (Cham: Springer, 2024).

10 Heeffer, "On the Nature and Origin of Algebraic Symbolism."

11 Hay, ed., *Mathematics from manuscript to print, 1300-1600* (Oxford: Clarendon Press, 1988); Gärtner, *Johannes Widmanns 'Behende vnd hubsche Rechenung': die Textsorte 'Rechenbuch' in der Frühen Neuzeit* (Tübingen: Niemeyer, 2000); Rommevaux, Spiesser, and Massa Esteve, eds., *Pluralité de l'algèbre à la Renaissance* (Paris: Honore Champion, 2012).

12 Rommevaux, Spiesser, and Massa Esteve, eds., *Pluralité de l'algèbre à la Renaissance*; Cuomo, "Mathematical traditions in Ancient Greece and Rome," *HAU: Journal of Ethnographic Theory* 9/1 (2019): 75–85; Chemla, "Cultures of Computation and Quantification in the Ancient World: An Introduction," in *Cultures of Computation and Quantification in the Ancient World: Numbers, Measurements, and Operations in Documents from Mesopotamia, China and South Asia*, ed. by Chemla, Keller, Proust, 1–140 (Cham: Springer, 2022); Boucard and Morel, "New Ob-

history of origins and necessary progress, the history of mathematics appears in these accounts as a more complex – and in several cases more surprising – history of evolution of calculative practices.

Building on these approaches, and following a line initiated by Van Egmond,[13] to understand the roots of the distinctive characteristics of the form of algebra that emerged in late medieval Italy, we first need to focus on the socio-economic setting in which abacus mathematics appeared. In other words, we need to discuss the deep connections between the emergence of abacus mathematics and the so-called "commercial revolution of the thirteenth century": a period marked by rapid expansion in trade, finance, and urban enterprise.[14] This phase of economic transformation was characterised by the rise of new corporate structures (such as the *compagnia* and the *accomandita*), the introduction of new accounting practices (such as double-entry bookkeeping), and the development of financial instruments such as bills of exchange and marine insurance.[15] These innovations created an unprecedented demand for practical mathematical skills, which late medieval city-states satisfied with the foundation of vocational schools – the so-called "abacus schools" – which had the specific aim of securing the intergenerational transmission of a mathematics that had become central to commercial activity.[16]

Abacus manuals differ markedly from what we would today recognize as mathematical texts. Instead of providing theoretical expositions, these works consist largely of extensive collections of solved problems. These worked *ragioni* are typically grouped according to their domain of application – such as problems related to exchange, divisions of profits and losses, interest rates, barter, and related mercantile operations. A

jects, Questions, and Methods in the History of Mathematics," *Histories* 2/3 (2022): 341–351; Rowe and Dauben, eds., *A Cultural History of Mathematics*, 6 vols. (London: Bloomsbury Publishing, 2024).

13 Van Egmond, "The Commercial Revolution and the Beginnings of Western Mathematics in Renaissance Florence, 1300–1500 (PhD dissertation)," University Microfilms International 1976.

14 De Roover, "The Commercial Revolution of the Thirteenth Century," in *Enterprise and secular change: readings in economic history*, ed. by Lane and Riemersma, 80–85 (London: Allen and Unwin, 1953); Lopez, *The commercial revolution of the Middle Ages, 950-1350* (Cambridge; New York: Cambridge University Press, 1976).

15 Goldthwaite, *The economy of Renaissance Florence* (Baltimore, Md: Johns Hopkins University Press, 2009), 63–114.

16 Goldthwaite, "Schools and Teachers of Commercial Arithmetic in Renaissance Florence," *Journal of European Economic History* 1/2 (1972): 418–433; Grendler, *Schooling in Renaissance Italy: literacy and learning, 1300-1600* (Baltimore: Johns Hopkins University Press, 1989), 306–332; Ulivi, "Scuole e maestri d'abaco in Italia tra Medioevo e Rinascimento," in *Un ponte sul Mediterraneo: Leonardo Pisano, la scienza araba e la rinascita della matematica in Occidente*, ed. by Giusti and Petti, 121–159 (Firenze: Polistampa, 2002); Black, *Education and society in Florentine Tuscany* (Leiden; Boston: Brill, 2007); Masiero, *Literacy and learning in Latin and abacus schools in Verona (1405–1509)* (Leiden: Brill, 2026).

number of particularly sophisticated manuals also include sections devoted to algebra, demonstrating an early integration of advanced mathematical reasoning into a fundamentally practical framework. This organization renders abacus manuals more akin to recipe books than to systematic treatises. This mode of exposition aligns them with a broader genre of practical literature that appeared in increasing numbers from the late middle ages, which includes *pratiche di mercatura, books of arts, books of secrets, ricordanze,* and other instructional or technical compendia.[17] Such texts share a common orientation toward utility and application, privileging procedural knowledge over theoretical explanations.

The broader commercial environment and the specific context of abacus schools in which these texts appeared helps illuminate the emergence of a form of mathematical knowledge characterized by distinct criteria for mathematical validity and standards of practice. For instance, Høyrup has recently noted how the competitive environment in which abacus masters operated encouraged them to conceptualise mathematical knowledge primarily in terms of a body of techniques for *solving problems*:

> Abbacus masters were members of a liberal profession in a market economy … They competed with each other, either for students or for positions at municipally financed abbacus schools. There, ability to solve problems proposed by the adversary at a competition or the aptitude to propose problems the adversary could not solve were useful … Even school students had to be trained in solving problems – simple problems in their case. Competition as well as abbacus-school teaching pushed toward seeing problem solution as the general format in which mathematical knowledge should be formulated.[18]

Such contextual considerations are crucial for understanding why the seeds of what would eventually become European symbolic algebra arose not within the tradition of learned mathematics pursued in late medieval universities, but within the practical, problem-oriented environment of commercial arithmetic. A useful point of departure for clarifying the nature of this transformation is offered by Mahoney's (1980) influential characterisation of the "algebraic mode of thought," that emerged in modern European algebra:

> What should be understood as the "algebraic mode of thought"? It has three main charac-

[17] Leong, *Recipes and everyday knowledge: medicine, science, and the household in early modern England* (Chicago; London: University of Chicago Press, 2018); Creager, Grote, and Leong, "Learning by the book: manuals and handbooks in the history of science," *BJHS Themes* 5 (2020): 1–13; Smith, *From lived experience to the written word: reconstructing practical knowledge in the early modern world* (Chicago: University of Chicago Press, 2022).

[18] Høyrup, *The World of the Abbaco*, 113–114.

> teristics: first, this mode of thought is characterized by the use of an operative symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates. Second, precisely because of the central role of combinatory operations, the algebraic mode of thought deals with mathematical relations rather than objects. Third, the algebraic mode of thought is free of ontological commitment … In particular, this mode of thought is free of the intuitive ontology of the physical world. Concepts like "space," "dimension," and even "number" are understood in a purely mathematical sense, without reference to their physical interpretation.[19]

While Mahoney used this definition to describe modern European algebra, the seeds of this "mode of thought" can already be observed in the late medieval tradition of practical arithmetic. The adoption of Indo-Arabic numerals satisfied the fundamental precondition for the first characteristic listed by Mahoney – that is, the use of an operative symbolism. This is because Indo-Arabic numerals provide a numerical notation that is inherently operative, as the algorithms to carry out calculations with this notation require the practitioner to operate with the signs themselves.

Moreover, as Mahoney observed, the development of such a mode of thought presupposes a shift away from the classical, Euclidean ontology that long underpinned mathematical reasoning in the learned tradition. In the mercantile environment of the thirteenth-century commercial revolution, would-be merchants – i.e., the main constituency of students who attended abacus schools – required a set of mathematical tools to solve practical problems rather than a theoretical mathematical system. This environment favoured the emergence of a body of mathematical knowledge that broke decisively with the intuitive, physically grounded conception of mathematics characteristic of the Euclidean paradigm, and the development of an operative mathematical language.

Nevertheless, the absence of theoretical demonstrations in abacus manuals does not mean that there is no proof for their mathematics; rather, it means that their proof is to be found elsewhere. Merchants, bankers, and other practitioners prioritised the *effectiveness* of a rule over its thorough, geometrical demonstration. The mathematical knowledge included in abacus manuals reflects this prioritization of the application over the demonstration. Instead of being demonstrated deductively, the validity of the rules presented in abacus manuals was *corroborated* through successful application in long lists of worked examples. Rather than theorems, we should think of the rules included in abacus manuals

19 Mahoney, "The Beginnings of Algebraic Thought in the Seventeenth Century," in *Descartes: Philosophy, Mathematics and Physics*, ed. by Gaukroger, 141–155, 142 (Brighton, Sussex: The Harvester Press, 1980). The paper was originally published as Mahoney, "Die Anfänge der algebraischen Denkweise im 17. Jahrhundert," *RETE: Strukturgeschichte der Naturwissenschaften* 1 (1971): 15–31.

in terms of mathematical tools. The mathematics of these texts is thus best understood as a procedural and operational mathematical craft – rather than as a deductive system – whose validity is justified through use, in what Field has characterised as a form of "quasi experimental proof."[20] This explains the distinctive recipe-like structure of these texts, composed predominantly of solved examples framed in the conditional form "if we are given this problem, *this* is what we need to do."

Because the attention of these mathematical practitioners lay on solution rather than demonstration, abacus mathematics enjoyed a degree of exploratory freedom rarely available to mathematicians working in the wake of the Euclidean model. This freedom is also reflected in the flexible and selective manner in which abacus texts engaged with earlier sources, because they reveal a highly selective reception of prior mathematical works. A clear example of this is provided by the oldest abacus manual extant – the so-called *Livero de l'abbecho* (Florence, Biblioteca Riccardiana, Ricc. 2404) – whose text is the result of several stratifications. The text includes sections whose sources are not attested, and sections that are a vulgarization of the Latin text of Leonardo Pisano's *Liber abaci*. These latter sections, however, are not a simple vulgarization of Fibonacci's text. Rather, as Andrea Bocchi has shown, these sections are the outcome of a series of previous manipulations and reorganisations of their source.[21]

Similarly, the sections that do not derive from Fibonacci's work – hence, texts for which the *Livero* provides the first known source – reappeared with various modifications in several subsequent abacus manuals. For example, while Fibonacci introduced the rule of three with a thorough and systematic discussion,[22] the formula used in the *Livero* prior-

20 Field, "The Unhelpful Notion of 'Renaissance man'," *Interdisciplinary Science Reviews* 41/2–3 (2016): 188–201, 197–198.

21 Bocchi, ed., *Lo livero de l'abbecho* (Pisa: Edizioni ETS, 2017), 36–37.

22 Fibonacci discusses the rule of three in the eighth chapter of his *Liber abaci*. After two chapters outlining the possible combinations of wares, units, weights, and measures, Leonardo Pisano provides the following illustration of the rule, which is substantially more thorough than the concise formula of the *Livero*: "Sed primum ostendam unde hic modus procedit. Sunt enim, ut dixi, in negotiationibus quattuor numeri proportionales, scilicet ut sicut primus est ad secundum ita tertius ad quartum, hoc est sicut numerus alicuius quantitatis mercis est ad numerum quantitatis sui pretii, ita numerus cuiusvis quantitatis eiusdem mercis est ad numerum sui pretii; vel sicut aliqua quantitas cuiusvis mercis est ad quamvis quantitatem eiusdem mercis, ita est pretium unius ad pretium alterius. Et cum ita quattuor quantitates proportionales sunt, erit multiplicatio secunde in tertiam equa multiplicationi prime in quartam, ut in arismetricis et in geometria probatum est. Quare si quarta quantitas est ignota tantum, ex multiplicatione quidem secunde quantitatis in tertia divisa per primam, nimirum ex divisione quarta quantitas provenit; quia cum dividitur aliquis numerus per aliquem numerum et ex divisione aliquid proveniat, si proveniens in divisorem multiplicaveris, nimirum divisus numerus inde proveniet. Similiter si tertia quantitas ignoratur, dividenda est per secundam multiplicatio prime in quartam." See Giusti and D'Alessandro, eds., *Leonardi Bigolli Pisani Liber Abbaci*, 141–142.

itises memorability over mathematical thoroughness, reflecting the didactic character of abacus manuals:

> Lo p(r)imo chapitolo è-ne de le reg(o)le d(e) le tre chose.
> Se ce fosse dicta alchuna ragione e · lla q(ua)-
> le se proponesse tre chose, sì devemo m(ultip)licare
> quilla chosa che noie volemo sap(er)e (contra) q(ui)lla
> che non è de quilla medessme, a pa(r)ti(r)e nell'a-
> ltra.[23]

This formulation of the rule is attested in several subsequent abacus manuals with various modifications. These manuals include, among others, the so-called *Tractatus algorismi* by Jacopo da Firenze (Florence, Biblioteca Riccardiana, Ricc. 2236; Rome, Biblioteca Vaticana, Vat. Lat. 4826, Milan, Biblioteca Trivulziana, 90), the *Liber habaci* attributed to Paolo Gherardi (Florence, Biblioteca Nazionale Centrale, Magl. CI. XI. 88), the *Trattato di tutta l'arte dell'abacho* attributed to Paolo dell'Abaco (Florence, Biblioteca Nazionale Centrale, Fond. Naz. II. IX. 57, and several copies), and the *Trattato d'abaco* by Piero della Francesca (Florence, Biblioteca Mediceo-Laurenziana. Ash 359 (291)). Similar formulations of the rule of three also appeared in printed texts, including the so-called *Aritmetica di Treviso* (1478) and Luca Pacioli's *Summa* (1494). Across these texts, the formula first presented in the *Livero* was transmitted with various modifications that reflect, rather than the faithful reproduction of an authoritative model, what Bocchi has described as a form of "professional memory."[24]

The above-mentioned *Trattato di tutta l'arte dell'abacho* attributed to Paolo dell'Abaco is another example of the weak authoriality of these sources. A complete copy is included in the oldest manuscript preserving the text: the manuscript Fond. Naz. II. IX. 57 of the Biblioteca Nazionale Centrale of Florence. More than half of the other manuscripts preserving the text (six out of ten) include either partial copies of the text or excerpta.[25] In the fifteenth century, the *Trattato d'abacho* by Maestro Benedetto was a particularly successful text, with over fifteen manuscript copies surviving to this day. Most of these copies only include parts from Benedetto's text, which were probably selected following the needs and interests of the users of these texts.[26]

23 Bocchi, ed., *Lo livero de l'abbecho*, 163.
24 Ibid., 65.
25 Ceccherini, "Come nasce un libro d'abaco. Struttura, tradizione e storia del ms. Firenze, Biblioteca nazionale centrale, II.IX.57," *Scrineum* 22/1 (2025): 1–79, 58–59.
26 Van Egmond, *Practical mathematics in the Italian Renaissance*, 356; Danna, *The Craft of Indo-Arabic Numerals: How Practical Arithmetic Shaped Commerce and Mathematics in Western Europe, 1200-1600* (Cambridge, MA: Harvard University Press, 2026), 148–149.

As these examples show, rather than transmitting earlier material verbatim, the writers of abacus manuals adapted, modified, and reorganized previous abacus works with a degree of freedom comparable to how artisans would adapt, depending on convenience, tools and practices inherited from their peers. This adaptive and experimental attitude was central to the development of the operative mathematical language in which the seeds of European algebraic symbolism could be developed.

This environment – characterised by a degree methodological freedom and a limited emphasis on the rigour of mathematical demonstrations – created the conditions for the emergence of a form of algebra that experimented with new algorithms and symbolic practices. The different ontology of this mathematics is already signalled by the terms used to describe algebraic quantities: reflecting the Arabic origins of their algebra, abbacists called the unknown "thing" (*cosa*, derived from the Arabic *shay*) and its square "wealth" (*censo*, derived from the Arabic *māl*).[27] The most significant consequence of this shift in mathematical reasoning was the abandonment of geometrical methods in favour of operational and algorithmic ones. Whereas the algebraic sections of Fibonacci's *Liber abaci* had offered a unique synthesis of Arabic algebra and Euclidean-style geometric demonstrations, the algebraic sections of abacus manuals adopted a decisively non-geometric orientation. Their authors privileged algorithmic procedures over spatial reasoning, and the criteria for mathematical validity shifted accordingly. In place of thorough geometrical demonstrations, these texts relied on the successful application of a rule to a particular problem as the primary warrant of validity. What counted as proof, therefore, differed fundamentally from the expectations governing learned mathematical texts: it was not a matter of deductive demonstration, but of effective performance.

As noted by Franci and Toti Rigatelli, abacus manuals with algebraic sections exhibit these characteristics already from the fourteenth century:

> In Italy, little more than one century after the *Liber abaci*, algebra had already developed sufficiently to permit the authors a free and easy use of the calculation of rationalization processes, radicals, and algebraic fractions, while completely omitting, among other things, the geometric references so dear to Leonardo Pisano.[28]

An early example of abacus algebra can be found in Jacopo da Firenze's *Tractatus algorismi*, first written in Montpellier in 1307. The algebraic section of this text presents no

[27] These terms already appeared in al-Khwārizmī's algebraic work. On the reception of these mathematical techniques, see Moyon, "La restauration et la comparaison, ou l'art de résoudre des équations quadratiques dans l'Europe latine, *Revue d'Histoire des Mathématiques* 23/2 (2017): 233–299; Sammarchi, "Which al-Khwārizmī? Divergent Scholarly Conceptions of a Canonical Figure," *Arabic Sciences and Philosophy* (forthcoming).

[28] Franci and Rigatelli, "Fourteenth century Italian algebra," 29.

geometric proofs, and lists solution formulas for twenty equation forms, including some cubic and quartic equations that can be solved by reducing them to a lower degree. Instead of being introduced by a theoretical premise that demonstrates their validity or explains their technical terms, these rules are simply stated, and are subsequently applied to the solution of a number of practical problems.[29]

While the precise date of the algebraic section of Jacopo da Firenze's *Tractatus algorismi* is debated,[30] we find an unquestionably early fourteenth-century example of abacus algebra in the *Libro di ragioni* written by Paolo Gherardi in 1328, another Florentine active in Montpellier. This text includes fifteen solution rules to linear, quadratic, and cubic equations, six of which had already appeared in al-Khwārizmī's and Fibonacci's works. As in the previous example, these rules were not introduced by a theoretical premise, were not supported by geometric diagrams, and were directly applied to the solution of practical problems. Most interestingly, some of Paolo Gherardi's rules were not valid. The presence of false rules is particularly interesting, because it reflects the high degree of experimentalism that characterised the early tradition of abacus algebra.[31] Other early abacus texts that include a treatment of algebra, such as the Manuscript Ricc. 2252 of the Biblioteca Riccardiana of Florence and the Manuscript 1754 of the Biblioteca Statale of Lucca, presented and rearranged the rules of previous algebraic texts.[32]

The first mathematical text entirely dedicated to algebra was probably written in Venice in the 1340s. The so-called *Aliabraa argibra* is attributed to a maestro Dardi da Pisa, and includes 198 rules, four of which for the solution of cubic and quartic equations. The author did not know a general method for solving such equations, and could handle only those with particular combinations of coefficients. Yet this limitation itself underscores the experimental character of the tradition of abacus mathematics, and its ongoing exploration of algorithmic and algebraic procedures.[33] Starting from a survey of the algebraic

29 Franci, "The History of algebra in Italy in the 14th and 15th centuries: some remarks on recent historiography," 179–180; Høyrup, *The World of the Abbaco*, 198–209; 432–433.

30 The date of the algebraic section of the text is debated. See Høyrup, *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture* (Basel: Birkhäuser, 2007); Van Egmond, "Jacopo da Firenze's Tractatus Algorismi edited by Jens Høyrup," *Aestimatio: Critical Reviews in the History of Science* 6 (2009): 37–47; Oaks, "Essay review: Medieval Italian practical mathematics," *CHSPM/SCHPM Bulletin* (2009); Høyrup, "A response to Van Egmond on Høyrup, Jacopo da Firenze's Tractatus Algorismi," *Aestimatio: Critical Reviews in the History of Science* 6 (2009): 116–126.

31 Franci, "The History of algebra in Italy in the 14th and 15th centuries: some remarks on recent historiography," 178–179; Høyrup, *The World of the Abbaco*, 209–215.

32 Franci, 'The History of algebra in Italy in the 14th and 15th centuries: some remarks on recent historiography," 181–182.

33 Van Egmond, "The algebra of master Dardi of Pisa," *Historia Mathematica* 10/4 (1983): 399–421; Franci, ed., *Maestro Dardi (s. xiv) Aliabraa Argibra (ms. I.VII.17 Biblioteca Comunale*

contents of these early texts, Van Egmond suggested a possible classification of their rules into different "families," and noted that they were generally characterised by accretional and incremental experimentation. He argued that the distinctive "pattern" of this tradition consisted of "direct copying mixed with reordering and the addition of new terms and cases, so that overall [one finds] a steady increase in the highest power, number of terms, and total number of equations."[34]

The fifteenth century is the period in which these sprawling experimentations started to be selected and systematised. This is clearly exemplified by the so-called "abacus encyclopaedias" written in this period, such as the ms L.IV. 21 of the Biblioteca Comunale of Siena, the ms. Pal. 573 of the Biblioteca Nazionale Centrale of Florence, and the ms. Ottobon. Lat. 3307 of the Biblioteca Apostolica Vaticana. These large and precious manuscripts comprise lengthy and encyclopaedic treatments of abacus mathematics, and reflect the changing use of abacus texts around the middle of the 15$^{th}$ century, as they were probably designed as collectable objects for prestigious libraries. These manuscripts also include algebraic sections, where one can find a critical selection of the algebraic rules attested in the previous centuries. For example, these texts do not feature Paolo Gherardi's invalid rules, and present a selection of algebraic rules that are attributed to abacus masters of the previous generations.[35]

The fifteenth century is also the period in which we find evidence of experimentation in mathematical notation. These were again carried out by practitioners who had either had their foundational mathematical training, or who wrote themselves works of practical arithmetic. The former case is exemplified by Giovanni Bianchini, a merchant and administrator with an interest in astronomy, whose astronomical tables display remarkably early experimentation with decimal notation.[36] The latter case is exemplified by the famous artist Piero della Francesca. In addition to being among the most important painters of the fifteenth century, Piero della Francesca was also an accomplished writer of mathematical texts. His *Trattato d'abaco* – which was probably among his earliest works – falls squarely into the tradition of practical arithmetic, and includes a thorough algebraic section.[37] This

*di Siena)*; Wagner, "Mordekhai Finzi's translation of Maestro Dardi's Italian Algebra, a Partial Edition," in *Latin-into-Hebrew: Texts and Studies. Volume 2*, ed. by Fontaine and Freudenthal, 195–212; 437–501 (Leiden; Boston: Brill, 2013).

34 Van Egmond, "The Study of Higher-Order Equations in Italy before Pacioli," 306.

35 Franci, "The History of algebra in Italy in the 14th and 15th centuries: some remarks on recent historiography," 187–188. For a discussion of the algebraic contents of these manuscripts, see Høyrup, *The World of the Abbaco*, 267–304.

36 Van Brummelen, "Decimal fractional numeration and the decimal point in 15th-century Italy," *Historia Mathematica* 66 (2024): 1–13.

37 Piero della Francesca's *Trattato d'abaco* is published in Arrighi, ed., *Piero della Francesca. Trattato d'abaco: dal Codice Ashburnhamiano 280, 359*-291*, della Biblioteca medicea laurenziana di Firenze* (Pisa: Domus Galilaeana, 1970); Dalai Emiliani, Besomi, Maccagni, Gamba, Montebel-

text includes sixty-one algebraic formulas for solving specific forms of quadratic, cubic, quartic, and quintic equations, most of which draw heavily on earlier algebraic manuals, and displays an experimental symbolic notation for the powers of the unknown.[38] Specifically, Piero used geometrical signs to represent the powers of the unknown: he represented the first degree of the unknown with a superscript "–"; the square of the unknown with a superscript square "□"; the cube of the unknown with a superscript "*c*" or "Δ"; and the fourth degree with a double square "□□." Hence, Piero wrote what we would represent as $5x$ as "$5^{-}$"; $5x^2$ as "$5\mathrm{x}^{\square}$"; $5x^3$ as "$5x^{c}$" or "$5x^{\Delta}$"; and $5x^4$ as "$5x^{\square\square}$." While experimenting with this symbolism, Piero also used the standard names given to the degrees of the unknown in other abacus texts, i.e. *cosa*, *censo*, *cubo*, and *censo di censo*.[39]

Luca Pacioli's *Summa de Arithmetica Geometria Proportioni et Proportionalità*, published in Venice in 1494, closed the fifteenth century by making an encyclopaedic mathematical work appear in print. Moreover, by noting that "thus far" no general rules for the third and fourth degree equations had been identified – and by considering the possibility that these might be found in the near future – Pacioli anticipated the important developments of the following decades. The process through which the general solutions to cubic and quartic equations were found was highly influenced by the methods and experimentations described above. The process that led to this discovery – the first major mathematical breakthrough since antiquity – involved in different degrees Scipione dal Ferro, Antonio Maria Fior, Niccolò Tartaglia, Girolamo Cardano, and Ludovico Ferrari.[40] While their social profiles were not identical, these were all mathematicians who hybridized elements of the practical and the learned tradition, and were active in different degrees between the world of practical arithmetic, the court, and the university.

Scipione dal Ferro was the first to identify a solution formula for the equation that we would now write as $x^3 + px = q$. As far as we know, he never published this result, and only shared it with pupils and close relatives. The likely reason for this is that monopoly over this formula gave Scipione dal Ferro a competitive advantage during mathematical challenges, allowing him to consolidate a strong reputation, and to be appointed in the Bolognese *studium* with increasing salaries – even though with a chair of lower status than that of astronomy and other established disciplines.[41] A pupil of Dal Ferro's, An-

li, Derenzini, Mattesini, Valerio, and Sorci, eds., *Piero della Francesca. Trattato d'abaco*, III vol. (Roma: Istituto poligrafico e zecca dello stato, 2012).

38 Giusti, "Fonti medievali dell'Algebra di Piero della Francesca," *Bollettino di Storia delle Scienze Matematiche* 13/2 (1993): 199–250; Field, *Piero Della Francesca: a mathematician's art* (London, New Haven: Yale University Press, 2005).

39 Arrighi, ed., *Trattato d'abaco*, 12–13.

40 Toscano, *The Secret Formula: How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation* (Princeton, NJ: Princeton University Press, 2020).

41 Bianca, "Dal Ferro, Scipione."

tonio Maria Fior was one of the few people with whom the Bolognese mathematician shared his solution method.

In 1535, Fior challenged Niccolò Tartaglia to solve a set of mathematical problems. While not much about his life is known with certainty, Tartaglia was probably active as an abacus master, and was the author of several innovative works of vernacular mathematics.[42] Spurred on by the challenge posed by Fior, Tartaglia understood that the problems Fior had submitted to him could be reduced to the same equation form, and worked out by himself the solution formula. He thereby won the challenge, and made public the existence of a thus far unknown algebraic formula. News spread out and reached the attention of Girolamo Cardano around 1539. The illegitimate son of a Milanese doctor, Cardano graduated in medicine from the university of Padua, was active as a teacher of arithmetic in Milan, and managed to have important connections with the court.[43] Cardano's *Practica arithmeticae* had just been published, consolidating his standing as a distinguished mathematician and his connections with the Milanese court. Cardano started corresponding with Tartaglia, and – probably exploiting his ties with the court – managed to persuade Tartaglia to at least partially share his discovery with him. In exchange, Cardano promised never to reveal Tartaglia's discovery.

Cardano, however, suspected that Tartaglia knew more than he had shared, and hence continued to work on Tartaglia's formula with the assistance of his pupil Ludovico Ferrari. While also active as a practical mathematician and surveyor, Ferrari eventually managed to be appointed at the chair of arithmetic of the Bolognese *studium*, and died in unclear circumstances.[44] Together with finding out about the original discovery of the formula by Scipione dal Ferro, in their research Cardano and Ferrari were able to find the general solution to both cubic and quartic equations. In 1545, these results were made public for the first time in Cardano's *Ars Magna*. Cardano credited the discovery of the initial formula to both Dal Ferro and Tartaglia, but the latter saw this publication as a breach of the vow, and openly denounced Cardano for appropriating his work.[45]

While this opened one of the most bitter mathematical disputes of the sixteenth century, it is important to note that Tartaglia and Cardano had much that united them, as they were both heirs of the practical tradition of abacus mathematics. While their social profiles differed, they embodied the increasingly diverse social standing of the practical

42 Nenci, "Tartaglia, Niccolò."

43 Grafton, *Cardano's cosmos: the worlds and works of a Renaissance astrologer* (Cambridge, MA: Harvard University Press, 1999); Baldi and Canziani, eds., *Girolamo Cardano. Le opere, le fonti, la vita* (Milano: Franco Angeli, 1999).

44 Fiocca, "Alcune opere inedite di Ludovico Ferrari," *Bollettino di Storia delle Scienze Matematiche* VIII/2 (1988): 239–305.

45 Gabrieli, *Nicolò Tartaglia: invenzioni, disfide e sfortune* (Università degli studi di Siena, 1986); Toscano, *The Secret Formula.*

mathematician, as they both engaged with the traditional role of teachers of arithmetic as well as with diverse roles, including appointments in private families, at the university, and the court. This is also reflected in the kind of mathematics which they both practiced. Tartaglia actively contributed to expanding the field of application of abacus mathematics. His *Nova scientia*, published in 1537, is the first printed mathematical work dealing with artillery and, by providing some of the earliest experiments in formalising the motion of projectiles, reflects the increasing diffusion of firearms. Tartaglia never used geometrical methods in his algebraic texts, thereby fully adhering to the model of abacus algebra.

Cardano's works were also founded in the tradition of practical arithmetic, though in a more nuanced way. His *Practica arithmeticae* aimed at distinguishing between mathematical theory and its applications, but was clearly rooted in the tradition of abacus mathematics and its practical applications.[46] This is reflected in his Latin language, as Cardano was compelled to Latinize terms that had previously existed only in the vernacular, in some cases with rather amusing results. For example, Veronica Gavagna has noted the term *schisatio*, with which Cardano clearly latinised the Italian *schisare*, a technical term which abacus masters used to indicate the simplification of a fraction to its lowest terms – and which had no translation into Latin.[47]

This discussion brings us to the role of the rediscovery of classic mathematical works which unfolded in the same period and interacted in significant ways with these developments. By writing in Latin, Cardano aimed not only to make his mathematics accessible to a European audience, but also to give it a higher cultural standing. His "great art" (*ars magna*) exemplifies the hybridization of vernacular and learned mathematics, as Cardano rendered in Latin the language and methods of practical arithmetic. This hybridization of classical models and practical traditions is even more evident in Cardano's *Ars magna*. In his processes, Cardano coupled the algorithmic methods characteristic of abacus mathematics with geometrical demonstrations, clearly following the Euclidean model.[48]

However, when it came to demonstrations for which there is no straightforward geometrical representation (such as negative numbers, or quartic equations), Cardano had to resort to operative and algorithmic methods. For example, Cardano's "mercantile interpretation" of negative numbers (i.e. treating them as debts) reveals a clear influence

46 Gavagna, "Alcune osservazioni sulla Practica Arithmetice di Cardano e la tradizione abachistica quattrocentesca," in *Girolamo Cardano. Le opere, le fonti, la vita*, ed. by Baldi, Canziani, 273–312 (Milano: Franco Angeli, 1999).

47 Gavagna, "Medieval Heritage and New Perspectives in Cardano's Practica arithmetice," *Bollettino di storia delle scienze matematiche* XXX/1 (2010): 61–80, 67–68.

48 Girolamo Cardano, *Artis magnae, sive De regulis algebraicis, liber unus* (Nuremberg: Johannes Petreius, 1545), fols. 29v–49r. A modern edition of the text is published in *Girolamo Cardano. Artis magnae, sive De regulis algebraicis liber unus*, ed. by Tamborini (Milano: Franco Angeli, 2011), vol. xxv.

from the practices of abacus mathematics.[49] Likewise, Cardano managed to treat the so-called "irreducible case" of cubic equations, which implied working with the square root of negative numbers. Instead of resisting the possibility of square roots of negative numbers on the ground of the impossibility to represent them, Cardano called these numbers *quantitates sophisticae*, and manipulated them as any other number. In so doing, he followed the standards and practices of the generations of abacus masters before him, who had been tinkering with algorithmic methods without concerning themselves with their geometrical interpretation. At the same time, however, his use of geometrical demonstrations reflects Cardano's attempt to present algebra as a body of knowledge characterised by the same epistemological standing as classical mathematics. While he only wrote in the Italian vernacular, Tartaglia edited and commented the first Italian edition of Euclid' *Elements* in 1543, as well as a collection of writings by Archimedes. Both authors, hence, were in different ways aware of classical models, and their mathematics can be understood as an early example of the hybridizations of the two mathematical cultures that emerged in the sixteenth century.

The figure who best exemplifies this convergence, however, is Raphael Bombelli, whose *L'algebra parte maggiore dell'aritmetica* is probably the most advanced work of sixteenth-century Italian algebra. Printed in 1572 in Bologna, Bombelli's work aimed to define algebra as a distinct branch of mathematical theory. Bombelli was the son of a textile merchant, and was himself active as a practitioner, working as an engineer and architect. As shown by R. A. Jayawardene, however, while Bombelli's work had a marked theoretical character, its roots were clearly grounded in the practical tradition of abacus mathematics. A preparatory manuscript for the third book of Bombelli's *L'Algebra* preserved in the Biblioteca dell'Archiginnasio of Bologna, in fact, reveals that Bombelli's initial draft for the work used extensively the practical problems characteristic of abacus mathematics.[50] Bombelli decided to overturn the character of his text after he had the opportunity to study a manuscript preserved at the Biblioteca Vaticana which recorded a copy of Diophantus's "algebra" – a work which had been identified by Regiomontanus in the fifteenth century. The example of Diophantus's mathematics brought Bombelli to expunge all practical aspects from his algebra, and to present it as a worthy heir to what he perceived to be a classical model of mathematical purity.[51]

These examples show how the tools developed across a tradition that had first

[49] Gavagna, "Radices Sophisticae, Racines Imaginaires: The Origins of Complex Numbers in the Late Renaissance," in *The Art of Science*, ed. by Angelini, Lupacchini, 165–190, 166 (Cham: Springer, 2014).

[50] The text of the manuscript is edited in Fiocca and Leone, eds., *L'inedito terzo libro de L'Algebra di Rafael Bombelli* (Pisa: Edizioni della Normale, 2017).

[51] Jayawardene, "The Influence of Practical Arithmetics on the Algebra of Rafael Bombelli," *Isis* 64/4 (1973): 510–523.

emerged to solve problems that primarily concerned merchants and other practitioners were hybridised and repurposed in the sixteenth century. This transformation occurred together with the increasing social standing of the practical mathematician, who began to work beyond the abacus school and to appear in university rooms and courtly halls.[52] This transformation also occurred together with the increasing influence of a perceived model of classical mathematical rigour. This model, however, was grafted onto a substratum of a mathematical practices that had emerged in the previous two centuries to address practical problems. The classics exemplified an ideal of mathematical coherence and thoroughness, and practical mathematics provided an experimentalist and algorithmic model.

Original results in European mathematics developed at the convergence of these two paradigms: a "native" European model transmitted through scholarly practices, and a "foreign" model that originated in southern Asia, was introduced into western Europe through Mediterranean exchanges, and was first developed by practical mathematicians, until it eventually also influenced learned mathematical practices. The persistence of practical arithmetic techniques within foundational works of "modern" European algebra shows how the evolution of this mathematical knowledge was not a linear progression from "classical" to "modern" mathematics, but rather a complex synthesis of various traditions.

[52] Biagioli, "The Social Status of Italian Mathematicians, 1450–1600," *History of Science* 27/1 (1989): 41–95.

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